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SOME CHARACTERIZATIONS OF THE HOMOGENEOUS RULED REAL HYPERSURFACE IN A COMPLEX HYPERBOLIC SPACE

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Abstract. Motivated by the notion of ruled surfaces in Euclidean 3-space, we consider ruled real hypersurfaces isometrically immersed into an n-dimensional complex hyperbolic space of constant holomorphic sectional curvature $c(<0)$. The purpose of this expository paper is to give some geometric characterizations of the homogeneous ruled real hypersurface, that is, this ruled real hypersurface is an orbit of some subgroup of the full isometry group of the ambient space (see [2, 6, 7, 9, 10]).

1. Ruled real hypersurfaces

We study a real hypersurface M^{2n-1} (with Riemannian metric g) in a complex n-dimensional complex hyperbolic space $\mathbb{C}H^n(c), n \geqq 2$ of constant holomorphic sectional curvature $c(<0)$ through an isometric immersion. We first recall the definition of ruled real hypersurfaces M^{2n-1} in $\mathbb{C}H^n(c)$. A real hypersurface M is *ruled* if the holomorphic distribution $T^0M = \{X \in TM | g(X, \xi) = 0\}$ is integrable and all of whose leaves (i.e., maximal integral manifolds) are locally congruent to totally geodesic holomorphic hypersurfaces $\mathbb{C}H^{n-1}(c)$ in the ambient space $\mathbb{C}H^n(c)$, where ξ is the characteristic vector field with respect to the almost contact metric structure (ϕ, ξ, η, g) on M induced from the Kähler structure (g, J) of $\mathbb{C}H^n(c)$.

We recall the construction of ruled real hypersurfaces in $\mathbb{C}H^n(c)$. For a real smooth curve $\gamma = \gamma(s)$, $s \in \mathrm{I}$ parametrized by its arclength s, where I is on some open interval on $\mathbb{R}$, we take the totally geodesic holomorphic hypersurface $\mathbb{C}H^{n-1}_s(c)$ through the point $\gamma(s)$ in $\mathbb{C}H^n(c)$ in such a way that the holomorphic line spanned by $\dot\gamma(s)$ is perpendicular to the tangent space $T_{\gamma(s)}\mathbb{C}H^{n-1}_s(c)$. Then we get a ruled real hypersurface $M = \bigcup_{s\in \mathrm{I}} \mathbb{C}H^{n-1}_s(c)$.

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By the definition of ruled real hypersurfaces we find that a real hypersurface M is ruled in $\mathbb{C}H^n(c)$ if and only if $\widetilde{\nabla}_X Y \in T^0M$ for each $X, Y \in T^0M$, where $\widetilde{\nabla}$ denotes the Riemannian connection on $\mathbb{C}H^n(c)$. Then by simple calculation we know that a real hypersurface M is ruled in $\mathbb{C}H^n(c)$ if and only if $g(AX, Y) = 0$ for each $X, Y \in T^0M$, where A is the shape operator of M in $\mathbb{C}H^n(c)$.

We consider a subset $M_* = \{x \in M | \xi_x \text{ is not principal}\}$ of a ruled real hypersurface M. Note that M_* is open dense in M (see [8]). We define two functions $\mu = g(A\xi, \xi)$ and $\nu = \|A\xi - \mu\xi\|$ on M. Then M is decomposed as: $M = M_* \cup M_0 \cup M_\infty$, where $M_0 = \{x \in M | \nu(x) = 0\}$, $M_\infty = \{x \in M | \nu(x) = \infty\}$ and $M_* = \{x \in M | \nu(x) > 0\}$. Note that both of M_0 and M_∞ are subsets of measure zero on M (cf. [8]). In general, M has singular points, that is, M is not smooth on M_∞. So we omit M_∞ and consider a ruled real hypersurface M as a differentiable manifold $M = M_* \cup M_0$. By easy computation we see that the shape operator A of a ruled real hypersurface M in $\mathbb{C}H^n(c)$ is expressed on the set M_* as: $A\xi = \mu\xi + \nu U, AU = \nu\xi$ and $AX = 0$ for each $X (\in T^0M)$ perpendicular to U, where U is a unit vector defined by $U = (A\xi - \mu\xi)/\nu$.

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2. The homogeneous ruled real hypersurface in $\mathbb{C}H^n(c)$

We recall the classification of all *homogeneous* real hypersurfaces M^{2n-1} in $\mathbb{C}H^n(c)$, $n \geqq 2$ due to Bendt and Tamaru ([5]). Here, "homogeneous" means that these real hypersurfaces are orbits of closed subgroups of the full isometry group $\mathrm{I}(\mathbb{C}H^n(c))(= U(1, n))$ of the ambient space $\mathbb{C}H^n(c)$. By virtue of their works ([5, 4]) we can see that in the class of all homogeneous real hypersurfaces M of $\mathbb{C}H^n(c)$ M is minimal if and only if M is ruled in the ambient space. This fact implies that the homogeneous ruled real hypersurface is a nice example in the class of all homogeneous real hypersurfaces of $\mathbb{C}H^n(c)$.

Our aim here is to review the construction of the homogeneous ruled real hypersurface M in $\mathbb{C}H^n(c)$. For this purpose we first review the definition of circles in Riemannian geometry. Let $\gamma = \gamma(s)$ be a smooth real curve parametrized by its arclength s on a Riemannian manifold N with Riemannian metric g. If the curve γ satisfies the following ordinary differential equations with some nonnegative constant k:

$$\nabla_{\dot\gamma}\dot\gamma = kY_s \quad \text{and} \quad \nabla_{\dot\gamma}Y_s = -k\dot\gamma,$$

where $\nabla_{\dot\gamma}$ is the covariant differentiation along γ with respect to ∇ of N and Y_s is the so-called the unit principal normal vector of γ, we call γ a circle of curvature k on N. We regard a geodesic as a circle of null curvature. By virtue of the existence and the uniqueness of solutions to ordinary differential equations we can see that for each point $p \in N$, an arbitrary positive constant k and every pair of orthonormal vectors X and Y of T_pN, there exists locally the unique circle $\gamma = \gamma(s)$ on N satisfying the initial condition that $\gamma(0) = p, \dot\gamma(0) = X$ and $Y_0 = Y$.

The construction of the homogeneous ruled real hypersurface M is very simple. We take a circle $\gamma = \gamma(s)$ of curvature $\sqrt{|c|}/2$ on a totally real totally geodesic surface $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ in the ambient space $\mathbb{C}H^n(c)$. The circle γ is said to be horocyclic on $\mathbb{R}H^2(c/4)$ (for details, see [3]). Then the ruled real hypersurface M associated to the circle γ is homogeneous in $\mathbb{C}H^n(c)$. This construction shows that the full isometry group $\mathrm{I}(M)$ of our ruled real hypersurface M is a direct product of $\mathrm{I}(\mathbb{C}H^{n-1}(c))$ and a one-parameter subgroup $\{\varphi_s\}$ whose orbit is a circle of curvature $\sqrt{|c|}/2$ on $\mathbb{R}H^2(c/4)$. We remark that in the above decomposition $M = M_* \cup M_0 \cup M_\infty$ of the homogeneous ruled real hypersurface M we see that $M_* = \{x \in M | \nu(x) = \sqrt{|c|}/2\}$ and both of M_0 and M_∞ are empty sets (cf. [1]).

At the end of this section we recall the following which gives various characterizations of the homogeneous ruled real hypersurface in the class of all ruled real hypersurfaces in $\mathbb{C}H^n(c)$.

Proposition 1 ([1, 5, 8])**.** *For a ruled real hypersurface M in $\mathbb{C}H^n(c), n \geqq 2$, the following four conditions are mutually equivalent*:

(1) *M is homogeneous in $\mathbb{C}H^n(c)$;*
(2) *All principal curvatures of M are constant on M;*
(3) *The functions μ and ν satisfy $\mu \equiv 0$ and $\nu \equiv \sqrt{|c|}/2$;*
(4) *There exists an integral curve γ_ξ of the characteristic vector field ξ on M such that the curve γ_ξ is a circle of positive curvature $\sqrt{|c|}/2$ on a totally real totally geodesic plane $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ in the ambient space $\mathbb{C}H^n(c)$.*

3. Statements of results

We first pay attention to integral curves of the characteristic vector field ξ on the homogeneous ruled real hypersurface M. All of them are horocyclic on a totally real totally geodesic $\mathbb{R}H^2(c/4)$ in $\mathbb{C}H^n(c)$, so that they are congruent with each other up to $\mathrm{I}(\mathbb{C}H^n(c))$. Motivated by this fact we establish the following

Theorem 1 ([2])**.** *Let M be a ruled real hypersurface in $\mathbb{C}H^n(c), n \geqq 2$. Then M is homogeneous in the ambient space $\mathbb{C}H^n(c)$ if and only if M satisfies the following two conditions*:

(1) *Every integral curve γ_ξ of the characteristic vector filed ξ on M lies on a totally real totally geodesic surface $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ in $\mathbb{C}H^n(c)$.*

(2) *Every curve γ_ξ in Condition (1) has the first curvature function $\kappa_\xi = \|\widetilde{\nabla}_\xi \xi\|$ which does not depend on the choice of γ_ξ, where $\widetilde{\nabla}$ is the Riemannian connection of the ambient space $\mathbb{C}H^n(c)$. This means that for any integral curves γ_ξ^1 and γ_ξ^2 of ξ the first curvature functions κ_ξ^1 and κ_ξ^2 (of γ_ξ^1 and γ_ξ^2, respectively) satisfy $\kappa_\xi^1(s) = \kappa_\xi^2(s + s_0)$ for each s and some s_0.*

We next characterize the homogeneous ruled real hypersurface by studying the first curvatures of integral curves of the vector fields U and ξ.

Theorem 2 ([9]). *Let M^{2n-1} be a ruled real hypersurface of $\mathbb{C}H^n(c), n \geqq 2$. Then M is homogeneous in this ambient space if and only if M satisfies the following two conditions*:
(1) *Every integral curve γ_U of the vector field U has positive first curvature $\kappa_U := \|\widetilde{\nabla}_U U\|$ with $\kappa_U \leqq \sqrt{|c|}$ along the curve γ_U in $\mathbb{C}H^n(c)$;*
(2) *Every integral curve γ_ξ of the characteristic vector field ξ has positive first curvature $\kappa_\xi := \|\widetilde{\nabla}_\xi \xi\|$ with $\kappa_\xi \leqq \sqrt{|c|}/2$ along the curve γ_ξ in $\mathbb{C}H^n(c)$.*
Here, $\widetilde{\nabla}$ is the Riemannian connection of the ambient space $\mathbb{C}H^n(c)$.

We remark that all curves $\gamma_U = \gamma_U(s)$ in Condition (1) in Theorem 2 have the same curvature $\sqrt{|c|}$ on a holomorphic line $\mathbb{C}H^1(c)$ of $\mathbb{C}H_n(c)$ and all curves $\gamma_\xi = \gamma_\xi(s)$ in Condition (2) in Theorem 2 have the same curvature $\sqrt{|c|}/2$ on a totally real totally geodesic surface $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ in the ambient space $\mathbb{C}H^n(c)$.

Paying particular attention to the first curvature and the first holomorphic torsion of integral curves of the characteristic vector field ξ on the homogeneous ruled real hypersurface, we get the following

Proposition 2 ([6]). *Let M be a ruled real hypersurface in $\mathbb{C}H^n(c), n \geqq 2$. Then M is homogeneous in this ambient space if and only if every integral curve γ_ξ of the characteristic vector field ξ on M, considered as a curve in $\mathbb{C}H^n(c)$, satisfies the following two conditions*:
(1) *The first curvature $\kappa_\xi := \|\widetilde{\nabla}_\xi \xi\|$ of the curve γ_ξ is a constant function which is independent of the choice of γ_ξ, where $\widetilde{\nabla}$ is the Riemannian connection of $\mathbb{C}H^n(c)$;*
(2) *The first holomorphic torsion $\tau_{12} := g(\xi, JY_\gamma)$ is a constant function which is independent of the choice of γ_ξ, where J is the complex structure of $\mathbb{C}H^n(c)$ and Y_γ is the unit principal normal vector of the curve γ_ξ, so that Y_γ satisfies $\widetilde{\nabla}_\xi \xi = \kappa_\xi Y_\gamma$.*

The following theorem gives a necessary and sufficient condition for a ruled real hypersurface having constant mean curvature to be homogeneous in $\mathbb{C}H_n(c)$.

Theorem 3 ([7]). *Let M be a ruled real hypersurface of $\mathbb{C}H^n(c), n \geqq 2$. Then M is homogeneous in $\mathbb{C}H^n(c)$ if and only if M satisfies the following two conditions*:
(1) *M has constant mean curvature in $\mathbb{C}H^n(c)$, i.e.,* Trace A *is constant on M, where A is the shape operator of M in $\mathbb{C}H^n(c)$;*
(2) *Every integral curve γ_U of the vector field U has the first curvature function $\kappa_U = \|\widetilde{\nabla}_U U\|$ which does not depend on the choice of γ_U, where $\widetilde{\nabla}$ is the Riemannian connection of the ambient space $\mathbb{C}H^n(c)$. This means that for any integral curves γ_U^1 and γ_U^2 of U the first curvature functions κ_U^1 and κ_U^2 (of γ_U^1 and γ_U^2, respectively) satisfy $\kappa_U^1(s) = \kappa_U^2(s + s_0)$ for each s and some s_0.*

Observing integral curves of the characteristic vector field ξ on a ruled real hypersurface M, we obtain the following characterization of the homogeneous ruled real hypersurface in $\mathbb{C}H^n(c)$:

Theorem 4 ([10]). *For a ruled real hypersurface M in $\mathbb{C}H^n(c), n \geqq 2$ the following two conditions are equivalent*:

(1) *M is homogeneous in the ambient space $\mathbb{C}H^n(c)$;*
(2) *Every integral curve γ_ξ of the characteristic vector field ξ on M is mapped to a circle of the same positive curvature which is independent of the choice of γ_ξ in the ambient space $\mathbb{C}H^n(c)$.*

Note that in Theorem 4(2) a circle is not supposed to be lying locally on a totally real totally geodesic surface $\mathbb{R}H^2(c/4)$ in $\mathbb{C}H^n(c), n \geqq 2$.

The following result is a characterization of the homogeneous ruled real hypersurface in the class of *all* real hypersurfaces in $\mathbb{C}H^2(c)$.

Theorem 5 ([6])**.** *Let M be a real hypersurface in a complex hyperbolic plane $\mathbb{C}H^2(c)$. Then M is locally congruent to the homogeneous ruled real hypersurface in this space if and only if M satisfies the following two conditions*:
(1) *M has constant mean curvature, i.e.,* $\mathrm{Trace}\, A$ *is constant on M, where A is the shape operator of M in $\mathbb{C}H^2(c)$;*
(2) *Every integral curve γ_ξ of the characteristic vector field ξ on M is lying on $\mathbb{R}H^2(c/4)$ as a circle of the same curvature k which is independent of the choice of γ_ξ in $\mathbb{C}H^2(c)$.*

We finally pose the following open problem:

Problem 1. If we delete Condition (1) in Theorem 5, does this theorem hold true?

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