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# GEOMETRY OF GEODESIC SPHERES IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. A geodesic sphere G(r) of radius r  $(0 < r < \pi/\sqrt{c})$  of a complex projective space  $\mathbb{C}P^n(c)$  is one of the most interesting objects in differential geometry. This expository paper consists of two parts. In the first half, we study curve theory on G(r) (see [4, 9]). In the latter half, we investigate G(r) from the viewpoint of submanifold theory ([2, 11]).

# 1. INTRODUCTION

A geodesic sphere G(r) of radius r  $(0 < r < \pi/\sqrt{c})$  in an *n*-dimensional complex projective space  $\mathbb{C}P^n(c)$   $(n \ge 2)$  of constant holomorphic sectional curvature c(> 0)is impotant in intrinsic geometry as well as extrinsic geometry (i.e., submanifold theory).

In intrinsic geometry, for example it is known that G(r) is a naturally reductive Riemannian homogeneous manifold, so that every geodesic of G(r) is a homogeneous curve, namely it is an orbit of a one-parameter subgroup of the isometry group I(G(r)) of G(r) (cf. [12]). Moreover, when  $\tan^2(\sqrt{c} r/2) > 2$ , G(r) is a Berger sphere (see [15]). Inspired by these facts, we are interested in geodesics on G(r) of radius r ( $0 < r < \pi/\sqrt{c}$ ) in  $\mathbb{C}P^n(c)$ . We first investigate the length spectrum of G(r) in detail (see [4]). We next study non-geodesic homogeneous curves on G(r). We construct a family of closed non-geodesic homogeneous curves on G(r) with the same length by using an isometric embedding

(1.1) 
$$f \circ \iota_{G(r)} : G(r) \xrightarrow{\iota_{G(r)}} \mathbb{C}P^n(c) \xrightarrow{f} S^{n(n+2)-1}\left(\frac{n+1}{2n}c\right),$$

where  $\iota_{G(r)}$  is a natural inclusion mapping of G(r) into  $\mathbb{C}P^n(c)$  and f is so-called the first standard minimal embedding of  $\mathbb{C}P^n(c)$  into an (n(n+2)-1)-dimensional

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sphere  $S^{n(n+2)-1}((n+1)c/(2n))$  of constant sectional curvature (n+1)c/(2n) (see [9]).

In extrinsic geometry, again by using the above minimal embedding f we immerse each real hypersurface  $M^{2n-1}$  of  $\mathbb{C}P^n(c)$  into the ambient sphere  $S^{n(n+2)-1}((n+1)c/(2n))$  as follows:

(1.2) 
$$f \circ \iota_M : M \xrightarrow{\iota_M} \mathbb{C}P^n(c) \xrightarrow{f} S^{n(n+2)-1}\left(\frac{n+1}{2n}c\right),$$

where  $\iota_M$  is an isometric immersion of  $M^{2n-1}$  into  $\mathbb{C}P^n(c)$ . Note that the isometric immersion  $f \circ \iota_M$  does not have parallel second fundamental form for each real hypersurface M of  $\mathbb{C}P^n(c)$ . On the other hand, by direct computation we can see that  $f \circ \iota_M$  has parallel mean curvature vector in this sphere if and only if M is locally congruent to the geodesic sphere G(r) with  $\tan^2(\sqrt{c} r/2) = 2n+1$  in  $\mathbb{C}P^n(c)$ (cf. [11]). Needless to say, this geodesic sphere is a Berger sphere. Furthermore, it has an almost contact metric structure  $(\phi, \xi, \eta, \langle , \rangle)$ . In particular, when c = 8n+4, this geodesic sphere is a Sasakian space form of constant  $\phi$ -sectional curvature 8n + 5 (see [2]). These facts imply that for each of c(> 0) and  $n(\geqq 2)$ , every Ndimensional sphere  $S^N(\tilde{c})$  of constant sectional curvature  $\tilde{c}$  with  $(n+1)c/(2n) \ge \tilde{c}$ and N > n(n+2) - 1 admits a (2n-1)-dimensional Riemannian submanifold  $M^{2n-1}$  satisfying the following properties.

- (1) M is diffeomorphic but not isometric to a Euclidean sphere.
- (2) M is a homogeneous submanifold of the ambient sphere  $S^{N}(\tilde{c})$ , i.e., M is an orbit of some subgroup of the isometry group SO(N+1) of  $S^{N}(\tilde{c})$ . However, M is not a Riemannian symmetric space.
- (3) The mean curvature vector of M in  $S^N(\tilde{c})$  is nonzero-parallel with respect the normal connection of M.
- (4) M has an almost contact metric structure  $(\phi, \xi, \eta, \langle , \rangle)$ . Especially, in the case of c = 8n + 4, M is a Sasakian space form of constant  $\phi$ -sectional curvature 8n + 5.

In the latter half of this paper, we clarify these fundamental properties of a certain geodesic sphere G(r) in  $\mathbb{C}P^n(c)$ .

## 2. Length spectrum of geodesic spheres G(r) in $\mathbb{C}P^n(c)$

Let  $M^{2n-1}$   $(n \geq 2)$  be a real hypersurface with a unit nomal local vector field  $\mathcal{N}$ in  $\mathbb{C}P^n(c)$ . We denote by  $(\phi, \xi, \eta, \langle , \rangle)$  the almost contact metric structure of Minduced from the Kähler structure J of the ambient space  $\mathbb{C}P^n(c)$ . That is, this structure is defined by

$$\xi = -J\mathcal{N}, \ \eta(X) = \langle X, \xi \rangle$$
 and  $\phi X = JX - \eta(JX)\xi$  for all vectors X on M,

so that it satisfies

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1 \text{ and } \eta(\phi X) = 0 \text{ for arbitrary } X \text{ on } TM$$

We here recall the following fundamental equations for M, which are so-called Gauss formula and Weingarten formula, respectively.

(2.1) 
$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y \rangle \mathcal{N} \text{ and } \widetilde{\nabla}_X \mathcal{N} = -AX,$$

where  $\widetilde{\nabla}$  and  $\nabla$  are the Riemannian connections of  $\mathbb{C}P^n(c)$  and M, respectively, and A is the shape operator of M in  $\mathbb{C}P^n(c)$ . Then it follows from the fact that  $\widetilde{\nabla}J = 0$  and (2.1) that

(2.2) 
$$\nabla_X \xi = \phi A X$$

and

(2.3) 
$$(\nabla_X \phi) Y = \eta(Y) A X - \langle A X, Y \rangle \xi,$$

where X and Y are any vectors on M.

In the following, we consider a geodesic sphere G(r) of radius r  $(0 < r < \pi/\sqrt{c})$ in  $\mathbb{C}P^n(c)$ . So we use the commutative condition  $\phi A = A\phi$  without explanation. We recall the invariance  $\rho_{\gamma}$  for a geodesic  $\gamma = \gamma(s)$  on G(r), which is defined by  $\rho_{\gamma} = \langle \dot{\gamma}(s), \xi_{\gamma(s)} \rangle$  for  $-\infty < s < \infty$ . Equation (2.2) guarantees the constancy of  $\rho_{\gamma}$ with  $-1 \leq \rho_{\gamma} \leq 1$ .

$$\begin{aligned} \nabla_{\dot{\gamma}} \rho_{\gamma} &= \nabla_{\dot{\gamma}} \langle \dot{\gamma}, \xi \rangle = \langle \dot{\gamma}, \nabla_{\dot{\gamma}} \xi \rangle \\ &= \langle \dot{\gamma}, \phi A \dot{\gamma} \rangle = \langle \dot{\gamma}, A \phi \dot{\gamma} \rangle = \langle A \dot{\gamma}, \phi \dot{\gamma} \rangle \\ &= - \langle \phi A \dot{\gamma}, \dot{\gamma} \rangle = 0. \end{aligned}$$

This invariance  $\rho_{\gamma}$  is said to be the structure torsion of a geodesic  $\gamma$  on G(r). We shall state the congruence theorem on geodesics of G(r) in terms of their structure torsions. For this purpose we review fundamental notions on congruency for curves in Riemannian manifolds. Two curves  $\gamma_1, \gamma_2$  on a Riemannian manifold N are said to be congruent to each other in the usual sense if there exist an isometry of  $\varphi$  of N and a constant  $s_0$  satisfying  $\gamma_2(s) = (\varphi \circ \gamma_1)(s + s_0)$  for all s. In the case we can take  $s_0 = 0$ , they are said to be strongly congruent to each other. That is, we call two curves  $\gamma_1, \gamma_2$  on N strongly congruent to each other if there is an isometry  $\varphi$  of N with  $\gamma_2(s) = (\varphi \circ \gamma_1)(s)$  for all s. Trivially, a Riemannian manifold N is either a Euclidean space or a Riemannian symmetric space of rank one if and only if for every pair of geodesics  $\gamma_1, \gamma_2$  on N they are strongly congruent to each other. In this paper, we treat a curve on a Riemannian manifold N is a mapping of the real line  $\mathbb{R}$  into N.

**Lemma 1** ([4]). On a geodesic sphere G(r)  $(0 < r < \pi/\sqrt{c})$  of  $\mathbb{C}P^n(c)$ , two geodesics  $\gamma_1$ ,  $\gamma_2$  are strongly congruent to each other if and only if their structure torsions  $\rho_{\gamma_1}$ ,  $\rho_{\gamma_2}$  satisfy  $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$ .

We are now in a position to study lengths of closed geodesics of G(r). It suffices to consider the case of c = 4. Let  $\Pi : S^{2n+1}(1) \to CP^n(4)$  denote the Hopf fibration of a unit sphere. For a smooth curve  $\gamma$  on  $\mathbb{C}P^n(4)$  a smooth curve  $\tilde{\gamma}$  on  $S^{2n+1}(1)$ is called a horizontal lift of  $\gamma$  if  $\dot{\tilde{\gamma}}(s)$  is a horizontal vector and  $d\Pi(\dot{\tilde{\gamma}}(s)) = \dot{\gamma}(s)$  for all s. We note that a curve  $\gamma$  on G(r) is closed if and only if its horizontal lift  $\tilde{\gamma}$  on

 $S^{2n+1}(1)$  satisfies  $\tilde{\gamma}(s) = e^{i\theta}\tilde{\gamma}(s+s_0)$  with some constants  $\theta \in [0, 2\pi)$  and  $s_0(>0)$  for each  $s \in (-\infty, \infty)$ . The following elementary lemma is a key in our argument.

**Lemma 2** ([4]). Let  $\sigma$  be a smooth simple curve on  $\mathbb{C}P^n(4)$ . Suppose that a horizontal lift  $\tilde{\sigma}$  of  $\sigma$  on  $S^{2n+1}(1)$  is represented as

$$\tilde{\sigma}(s) = Ae^{\sqrt{-1}\,as} + Be^{\sqrt{-1}\,bs} + Ce^{\sqrt{-1}\,cs} + De^{\sqrt{-1}\,ds},$$

which is a curve in  $\mathbb{C}^{n+1}$  with nonzero vectors  $A, B, C, D \in \mathbb{C}^{n+1}$  and mutually distinct real numbers a, b, c, d satisfying a + b + c + d = 0 and  $a \neq 0$ . Then  $\sigma$  is closed if and only if all the ratios b/a, c/a, d/a are rational. In this case, its length is

length(
$$\sigma$$
) = 2 $\pi$  × L.C.M.  $\left\{\frac{1}{|b-a|}, \frac{1}{|c-a|}, \frac{1}{|d-a|}\right\}$ 

Here, for positive numbers  $\alpha_1, \alpha_2, \alpha_3$ , we denote by L.C.M.  $\{\alpha_1, \alpha_2, \alpha_3\}$  the minimum value of the set  $\{j\alpha_1|j=1,2,\ldots\} \cap \{j\alpha_2|j=1,2,\ldots\} \cap \{j\alpha_3|j=1,2,\ldots\}$ .

We remember that every geodesic of G(r)  $(0 < r < \pi/\sqrt{c})$  in  $\mathbb{C}P^n(c)$  is a homogeneous curve (see [12]), so that it is a simple curve. Then by Lemma 2 we obtain the following sufficient condition for a geodesic  $\gamma = \gamma(s)$  on G(r) to be closed, which can be written by its structure torsion  $\rho_{\gamma}$ :

**Theorem 1** ([4]). For a geodesic  $\gamma$  on a geodesic sphere G(r) of radius r (0 <  $r < \pi/\sqrt{c}$ ) in  $\mathbb{C}P^n(c)$  we have the following properties according to their structure torsions:

- (1) When  $\rho_{\gamma} = \pm 1$ , it is closed and its length is  $\pi \sin(\sqrt{c} r)$ ;
- (2) When  $\rho_{\gamma} = 0$ , it is also closed and its length is  $2\pi \sin(\sqrt{c} r/2)$ ;
- (3) When  $0 < |\rho_{\gamma}| < 1$ , it is closed if and only if its structure torsion  $\rho_{\gamma}$  is given by

$$\rho_{\gamma} = \frac{\pm q}{\sin(\sqrt{c} \ r/2)\sqrt{p^2 \tan^2(\sqrt{c} \ r/2) + q^2}}$$

with some relatively prime positive integers p and q with q .In this case, its length is

length(r) = 
$$2\delta(p,q)\pi\sqrt{(p^2\sin^2(\sqrt{c} r/2) + q^2\cos^2(\sqrt{c} r/2))/c}$$
.

Here,  $\delta(p,q)$  takes the value 2 when pq is even and takes the value 1 when pq is odd.

In consideration of Lemma 1 and Theorem 1 we find the following:

**Theorem 2** ([4]). On a geodesic sphere G(r)  $(0 < r < \pi/\sqrt{c})$  in  $\mathbb{C}P^n(c)$ , there exist countably infinite congruence classes of closed geodesics. The length spectrum  $\mathrm{LSpec}(G(r))$  of G(r) is a discrete unbounded subset in the real line  $\mathbb{R}$ .

We here investigate the first length spectrum  $\lambda_1$ , the second length spectrum  $\lambda_2$ and the third length spectrum  $\lambda_3$  of G(r) in detail.

**Proposition 1** ([4]). A geodesic sphere G(r)  $(0 < r < \pi/\sqrt{c})$  in  $\mathbb{C}P^n(c)$  has the following properties on the lengths of closed geodesics.

- (1)  $\lambda_1, \lambda_2$  and  $\lambda_3$  are simple, that is, each of their multiplicities is one.
- (2)  $\lambda_1 = (2\pi/\sqrt{c}) \sin(\sqrt{c} r)$ , which is the length of the geodesics of structure torsion  $\pm 1$ .
- (3) When  $0 < r \leq \pi/(2\sqrt{c})$ , we have  $\lambda_2 = (4\pi/\sqrt{c}) \sin(\sqrt{c} r/2)$ , which is the length of the geodesics with null structure torsion. When  $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$ , we have  $\lambda_2 = 2\pi/\sqrt{c}$ , which is the length of the geodesics with structure torsion  $\pm \cot(\sqrt{c} r/2)$ .
- (4) When  $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$  we have  $\lambda_3 = (4\pi/\sqrt{c}) \sin(\sqrt{c} r/2)$ , which is the length of the geodesics with null structure torsion. When  $0 < r \leq \pi/(2\sqrt{c})$  and it satisfies  $\sqrt{2k-1} \leq \cot r < \sqrt{2k+1}$  (k = 1, 2, ...), we have  $\lambda_3 = 2\pi\sqrt{\{4k(k+1)\sin^2(\sqrt{c} r/2)+1\}/c}$ , which is the length of the geodesics with structure torsion  $\pm 1/\left(\sin(\sqrt{c} r/2)\sqrt{(2k+1)^2}\tan^2(\sqrt{c} r/2)+1\right)$ .

We remark that the sectional curvature K of G(r)  $(0 < r < \pi/\sqrt{c})$  lies in the closed interval  $[(c/4) \cot^2(\sqrt{c} r/2), c + (c/4) \cot^2(\sqrt{c} r/2)]$ . Hence, as mentioned in Introduction, in the case of  $\tan^2(\sqrt{c} r/2) > 2$  we find that it is an example of a so-called Berger sphere. But for all lengths except the bottom  $\lambda_1$  of LSpec(G(r)), we find that the following inequality of Klingerberg's type holds.

**Corollary 1.** Except geodesics with structure torsion  $\pm 1$ , every geodesic  $\gamma$  of a geodesic sphere G(r)  $(0 < r < \pi/\sqrt{c})$  in  $\mathbb{C}P^n(c)$  satisfies length $(r) > 4\pi/\sqrt{c} (4 + \cot^2(\sqrt{c} r/2))$ .

Each element of LSpec(G(r)) is not necessarily simple. For example, for  $G(\pi/4)$  in  $\mathbb{C}P^n(4)$  we have

$$LSpec(G(\pi/4)) = \left\{ \pi, \sqrt{2} \ \pi, \sqrt{5} \ \pi, \sqrt{10} \ \pi, \sqrt{13} \ \pi, \sqrt{17} \ \pi, 5\pi, \sqrt{26} \ \pi, \sqrt{29} \ \pi, \sqrt{34} \ \pi, \sqrt{37} \ \pi, \sqrt{41} \ \pi, \sqrt{50} \ \pi, \sqrt{53} \ \pi, \sqrt{58} \ \pi, \sqrt{61} \ \pi, \sqrt{65} \ \pi, \sqrt{73} \ \pi, \ldots \right\}.$$

Though each element from  $\lambda_1 = \pi$  to  $\lambda_{16} = \sqrt{61} \pi$  is simple, we find that the multiplicity of  $\lambda_{17} = \sqrt{65} \pi$  is two. It is the common length of the geodesics with structure torsions  $3/\sqrt{65}$  and  $7/\sqrt{65}$ .

**Theorem 3** ([4]). For a geodesic sphere G(r)  $(0 < r < \pi/\sqrt{c})$  in  $\mathbb{C}P^n(c)$  we obtain the following:

- (1) If  $\tan^2(\sqrt{c} r/2)$  is irrational, every element of  $\operatorname{LSpec}(G(r))$  is simple.
- (2) If  $\tan^2(\sqrt{c} r/2)$  is rational, the multiplicity of each element of  $\operatorname{LSpec}(G(r))$  is finite, but not uniformly bounded and satisfies  $\limsup_{\lambda\to\infty} m(\lambda) = \infty$ . Its growth is less than polynomial growth. It satisfies  $\lim_{\lambda\to\infty} \lambda^{-\delta} m(\lambda) = 0$  for arbitrary positive  $\delta$ .
- (3) We denote by  $n(\lambda)$  the number of congruency classes of closed geodesics whose length is not longer than  $\lambda$ . Its growth is polynomial order of the second degree and satisfies  $\lim_{\lambda\to\infty} \lambda^{-2}n(\lambda) = 3cr/(4\pi^4 \sin(\sqrt{c} r))$ .

This theorem guarantees that on a geodesic sphere G(r)  $(0 < r < \pi/\sqrt{c})$  with irrational  $\tan^2(\sqrt{c} r/2)$  in  $\mathbb{C}P^n(c)$  two closed geodesics are congruent if and only if they have the same length. On the other hand, if  $\tan^2(\sqrt{c} r/2)$  is rational, we cannot classify congruency classes of closed geodesics only by their lengths.

## 3. Non-geodesic homogeneous curves of G(r) in $\mathbb{C}P^n(c)$

We are interested in finding a nice family of curves including all geodesics of  $G(r) (0 < r < \pi/\sqrt{c})$ . To do this, we recall the notion of Sasakian curves.

On a real hypersurface N in a Kähler manifold (M, J) a smooth curve  $\gamma$  is said to be a Sasakian curve if it satisfies

(3.1) 
$$\nabla_{\dot{\gamma}}\dot{\gamma} = k\phi\dot{\gamma}$$

with some constant k, where  $\phi$  is the structure tensor of N induced by J. Needless to say, for an arbitrary constant k and a unit vector v at each point  $x \in N$ , there exists the unique Sasakian curve  $\gamma$  satisfying (3.1) with initial condition that  $\gamma(0) = x$  and  $\dot{\gamma}(0) = v$ . Sasakian curves on a manifold admitting an almost contact metric structure can be considered as correspondences of Kähler circles on Kähler manifolds.

We here recall two invariances for a Sasakian curve  $\gamma = \gamma(s)$  on G(r) in  $\mathbb{C}P^n(c)$ . One is the structure torsion  $\rho_{\gamma} = \langle \dot{\gamma}(s), \xi_{\gamma(s)} \rangle$ . The other is the normal curvatue  $\kappa_{\gamma} = \langle A\dot{\gamma}(s), \dot{\gamma}(s) \rangle$ . By the same computation as above and (3.1) we find the constancy of  $\rho_{\gamma}$ :

$$\begin{aligned} \nabla_{\dot{\gamma}}\rho_{\gamma} &= \langle \nabla_{\dot{\gamma}}\dot{\gamma},\xi\rangle + \langle \dot{\gamma},\nabla_{\dot{\gamma}}\xi\rangle = k\langle\phi\dot{\gamma},\xi\rangle + \langle\dot{\gamma},\phi A\dot{\gamma}\rangle \\ &= \langle\dot{\gamma},\phi A\dot{\gamma}\rangle = \langle\dot{\gamma},A\phi\dot{\gamma}\rangle = -\langle\phi A\dot{\gamma},\dot{\gamma}\rangle = 0. \end{aligned}$$

Furthermore, by the fact  $\langle (\nabla_X A)X, X \rangle = 0$  for all vectors X on G(r), we see the constancy of  $\kappa_{\gamma}$ . We remark that for a Sasakian curve  $\gamma$  satisfying (3.1) on G(r) the first curvature  $\kappa_1 = \|\nabla_{\dot{\gamma}}\dot{\gamma}\|$  of  $\gamma$  is given by  $\kappa_1 = |k|\sqrt{1-\rho_{\gamma}^2}$ , so that it is constant along  $\gamma$ . Hence, in the following we say the constant k to be the *coefficient* of a Sasakian curve  $\gamma$  satisfying (3.1). As a matter of course we treat geodesics as Sasakian curves in a trivial sense.

For about Sasakian curves on G(r)  $(0 < r < \pi/\sqrt{c})$  in  $\mathbb{C}P^n(c)$ , the following is obtained in [3].

**Lemma 3.** Let  $\gamma_i$  (i = 1, 2) be Sasakian curves of coefficients  $\kappa_i$  and structure torsions  $\rho_{\gamma_i}$  on a geodesic sphere G(r)  $(0 < r < \pi/\sqrt{c})$  in  $\mathbb{C}P^n(c)$ . They are strongly congruent to each other if and only if one of the following conditions holds:

- i)  $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1;$
- ii)  $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$  and  $|\kappa_1| = |\kappa_2|$ ; iii)  $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$  and  $\kappa_1 \rho_{\gamma_1} = \kappa_2 \rho_{\gamma_2}$ .

As immediate consequences of this lemma we obtain the following corollary on the homogeneity of Sasakian curves on G(r).

**Corollary 2.** Every Sasakian curve on G(r)  $(0 < r < \pi/\sqrt{c})$  in  $\mathbb{C}P^n(c)$  is homogeneous. That is, it is an orbit of a one-parameter subgroup of the isometry group I(G(r)) of G(r).

In order to get geometric properties of Sasakian curves on a geodesic sphere G(r)  $(0 < r < \pi/\sqrt{c})$  in  $\mathbb{C}P^n(c)$ , we study them through the isometric embedding  $f \circ \iota_{G(r)}$  given by (1.1). As mentioned in Introduction, this embedding  $f \circ \iota_{G(r)}$  does not have parallel second fundamental form but is equivariant. So we can hence treat our geodesic sphere G(r) as a homogeneous submanifold in this sphere through this embedding  $f \circ \iota_{G(r)}$ .

Proposition 1 tells us that every integral curve  $\gamma$  of the characteristic vector field  $\xi$ on G(r) is the shortest closed geodesic for each radius  $r \in (0, \pi/\sqrt{c})$ . Furthermore, its shape  $\iota_{G(r)} \circ \gamma$  through the inclusion  $\iota_{G(r)} : G(r) \to \mathbb{C}P^n(c)$  is a Kähler circle of curvature  $\sqrt{c} \cot(\sqrt{c} r)$  in  $\mathbb{C}P^n(c)$ , namely this integral curve  $\gamma$  satisfies either  $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \sqrt{c} \cot(\sqrt{c} r)J\dot{\gamma}$  or  $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = -\sqrt{c} \cot(\sqrt{c} r)J\dot{\gamma}$ . If we see this curve through the embedding  $f \circ \iota_{G(r)}$ , we find the curve  $f \circ \iota_{G(r)} \circ \gamma$  is a small circle on a sphere  $S^{n(n+2)-1}((n+1)c/(2n))$  by the following lemma.

**Lemma 4** ([6]). A smooth curve  $\mu$  on  $\mathbb{C}P^n(c)$  is a Kähler circle of curvature  $\kappa$  if and only if the curve  $f \circ \mu$  on  $S^{n(n+2)-1}((n+1)c/(2n))$  is a circle of positive curvature  $\sqrt{\kappa^2 + ((n-1)c/(2n))}$ .

In this context, we naturally come to the position to pose the following problem:

Problem 1. Find and classify smooth curves  $\gamma$  on G(r) whose shape  $f \circ \iota_{G(r)} \circ \gamma$ through the equivariant isometric embedding  $f \circ \iota_{G(r)}$  are circles in  $S^{n(n+2)-1}((n+1)c/(2n))$ .

By virtue of Lemma 4 this problem is equivalent to the problem to find and to classify curves on G(r) which are mapped to Kähler circles in  $\mathbb{C}P^n(c)$  through the inclusion  $\iota_{G(r)}$ . For a smooth curve  $\gamma$  on G(r) we get by the Gauss formula that  $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma} + \langle A\dot{\gamma}, \dot{\gamma}\rangle \mathcal{N}$  and  $J\dot{\gamma} = \phi\dot{\gamma} + \rho_{\gamma}\mathcal{N}$ . We can hence obtain the following.

**Lemma 5** ([10]). A smooth curve  $\gamma$  on G(r) can be seen as a Kähler circle of curvature  $\kappa$  on  $\mathbb{C}P^n(c)$  through the inclusion  $\iota$  if and only if it satisfies both of the equations  $\nabla_{\dot{\gamma}}\dot{\gamma} = \pm\kappa\phi\dot{\gamma}$  and  $\langle A\dot{\gamma},\dot{\gamma}\rangle = \pm\kappa\rho_{\gamma}$ , where double signs take the same signatures.

By use of this lemma we can get the following answer to our problem. The answer depends on the radius of a geodesic sphere.

**Theorem 4.** Let G(r) be a geodesic sphere of radius  $0 < r \leq \pi/(2\sqrt{c})$  in  $\mathbb{C}P^n(c)$ .

- (1) For  $0 \leq k < c \{ \cot^2(\sqrt{c} r) + (n-1)/(2n) \}$ , there are no curves on G(r) whose shape through  $f \circ \iota_{G(r)}$  is a circle of curvature k on  $S^{n(n+2)-1}((n+1)c/(2n))$ .
- (2) When  $k^2 = c \{ \cot^2(\sqrt{c} r) + (n-1)/(2n) \}$ , the shape of a curve  $\gamma$  on G(r) through  $f \circ \iota_{G(r)}$  is a circle of curvature k if and only if it is a geodesic with structure torsion  $\rho_{\gamma} = \pm 1$ , which is an integral curve of  $\xi$  on G(r).

(3) When  $k^2 > c \{ \cot^2(\sqrt{c} r) + (n-1)/(2n) \}$ , the shape of a curve  $\gamma$  on G(r)through  $f \circ \iota_{G(r)}$  is a circle of curvature k if and only if it is a Sasakian curve of coefficient  $\pm \sqrt{k^2 - (n-1)c/(2n)}$  whose structure torsion is

$$\rho_{\gamma} = \pm c^{-1/2} \left\{ \sqrt{k^2 + (n+1)c/(2n)} - \sqrt{k^2 - (n+1)c/(2n)} \right\} \cot(\sqrt{c} r/2),$$

where double signs take the same signatures.

Trivially these curves in (2), (3) are closed with length  $2\pi/\sqrt{k^2 + (n+1)c/(2n)}$ .

**Theorem 5.** Let G(r) be a geodesic sphere of radius r with  $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$ in  $\mathbb{C}P^n(c)$ .

- (1) For  $0 \leq k < \sqrt{(n-1)c/(2n)}$ , there are no curves on G(r) whose shape thorough  $f \circ \iota_{G(r)}$  is a circle of curvature k on  $S^{n(n+2)-1}((n+1)c/(2n))$ .
- (2) When  $k = \sqrt{(n-1)c/(2n)}$ , the shape of a curve on G(r) through  $f \circ \iota_{G(r)}$  is a circle of curvature k if and only if it is a geodesic with structure torsion  $\rho_{\gamma} = \pm \cot(\sqrt{c} r/2).$
- (3) When  $\sqrt{(n-1)c/(2n)} < k < c \{ \cot^2(\sqrt{c} r) + (n-1)/(2n) \}$ , the shape of a curve  $\gamma$  on G(r) through  $f \circ \iota_{G(r)}$  is a circle of curvature k if and only if it is a Sasakian curve of coefficient  $\pm \sqrt{k^2 (n-1)c/(2n)}$  whose structure torsion is

$$\rho_{\gamma} = \pm c^{-1/2} \left\{ \sqrt{k^2 + (n+1)c/(2n)} - \sqrt{k^2 - (n+1)c/(2n)} \right\} \cot(\sqrt{c} \ r/2)$$
  
or

$$\rho_{\gamma} = \pm c^{-1/2} \left\{ -\sqrt{k^2 + (n+1)c/(2n)} - \sqrt{k^2 - (n+1)c/(2n)} \right\} \cot(\sqrt{c} r/2),$$

where double signs take the same signatures.

- (4) When  $k = c \{ \cot^2(\sqrt{c} r) + (n-1)/(2n) \}$ , the shape of a curve on G(r) through  $f \circ \iota_{G(r)}$  is a circle of curvature k if and only if it is a geodesic with structure torsion  $\rho_{\gamma} = \pm 1$ .
- (5) When  $k > c \{ \cot^2(\sqrt{c} r) + (n-1)/(2n) \}$ , the shape of a curve  $\gamma$  on G(r)through  $f \circ \iota_{G(r)}$  is a circle of curvature k if and only if it is a Sasakian curve of coefficient  $\pm \sqrt{k^2 - (n-1)c/(2n)}$  whose structure torsion is

$$\rho_{\gamma} = \pm c^{-1/2} \left\{ \sqrt{k^2 + (n+1)c/(2n)} - \sqrt{k^2 - (n+1)c/(2n)} \right\} \cot(\sqrt{c} r/2),$$

where double signs take the same signatures.

Trivially these curves in (2), (3), (4), (5) are closed with length  $2\pi/\sqrt{k^2 + (n+1)c/(2n)}$ .

- Remark 1. (1) For each curve  $\gamma$  in Theorems 4 and 5, the curve  $\iota_{G(r)} \circ \gamma$  is a homogeneous curve on totally geodesic  $\mathbb{C}P^1(c)(=S^2(c))$  of  $\mathbb{C}P^n(c)$  (see [3]). This fact shows that each curve in Theorems 4 and 5 is an orbit of a one-parameter subgroup of SO(3).
  - (2) For each Sasakian curve  $\gamma$  on G(r) the curve  $\iota_{G(r)} \circ \gamma$  is a homogeneous curve on totally geodesic  $\mathbb{C}P^2(c)$  of  $\mathbb{C}P^n(c)$  (see [1]). Hence every Sasakian curve on G(r) is an orbit of a one-parameter subgroup of SU(3).

- (3) Curves in Theorem 4(3), Theorem 5(3) and Theorem 5(5) are non-geodesic Sasakian curves.
- (4) There exist two non-geodesic Sasakian curves in Theorem 5(3) which are not congruent to each other with respect to I(G(r)), but they are mapped to a circle of the same curvature k on  $S^{n(n+2)-1}((n+1)c/(2n))$ . Hence these curves, considered as curves on this sphere, are congruent to each other with respect to the isometry group SO(n(n+2)) of the sphere.

# 4. Properties of certain geodesic spheres G(r) in $\mathbb{C}P^n(c)$ in submanifold theory

We first investigate the minimal embedding  $f: \mathbb{C}P^n(c) \to S^{n(n+2)-1}((n+1)c/(2n))$ which is defined by eigenfunctions of the first eigenvalue of the Laplacian  $\Delta$  on  $\mathbb{C}P^n(c)$ . The inner product of the first normal space of f is given by

(4.1) 
$$\langle \sigma_1(X,Y), \sigma_1(Z,W) \rangle = -(c/(2n))\langle X,Y \rangle \langle Z,W \rangle + (c/4) (\langle X,W \rangle \langle Y,Z \rangle + \langle X,Z \rangle \langle Y,W \rangle + \langle JX,W \rangle \langle JY,Z \rangle + \langle JX,Z \rangle \langle JY,W \rangle).$$

Here,  $\sigma_1$  denotes the second fundamental form of f. Equation (4.1) shows the following properties of f:

- i)  $f_1$  is minimal;
- ii)  $\sigma_1(JX, JY) = \sigma_1(X, Y)$  for  $\forall X, Y \in T\mathbb{C}P^n(c)$  (namely,  $\sigma$  is J-invariant);
- iii)  $\|\sigma_1(X,X)\| = \sqrt{(n-1)c/(2n)}$  for each unit vector X on  $\mathbb{C}P^n(c)$  (that is, f is  $\sqrt{(n-1)c/(2n)}$ -isotropic (cf. [14])).

We remark that  $\sigma_1$  is J-invariant is equivalent to saying that the second fundamental form  $\sigma_1$  of our embedding f is parallel. As mentioned in Introduction, the embedding f is usually called the first standard minimal embedding.

In this section, we immerse all real hypersurfaces M of  $\mathbb{C}P^n(c)$  into the sphere  $S^{n(n+2)-1}((n+1)c/(2n))$  (see (1.2)). Note that for every real hypersurface M, the second fundamental form of the isometric immersion  $f \circ \iota_M : M \to S^{n(n+2)-1}((n+1)c/(2n))$  is not parallel. However, in this class  $\{(M, f \circ \iota_M) | \iota_M : M \to \mathbb{C}P^n(c) \text{ is an isometric immersion}\}$  of all submanifolds in the sphere  $S^{n(n+2)-1}((n+1)c/(2n))$ , there exist nonzero-constant mean curvature submanifolds. For example, direct calculation tells us that the mean curvature  $H_r$  ( $0 < r < \pi/\sqrt{c}$ ) defined by the length of the mean curvature vector of the embedding  $f \circ \iota_{G(r)} : G(r) \to S^{n(n+2)-1}((n+1)c/(2n))$  given by (1.1) is expressed as

$$H_r^2 = \frac{c}{4(2n-1)^2} \left\{ (2n-1)^2 \cot^2\left(\frac{\sqrt{c}}{2}r\right) + \tan^2\left(\frac{\sqrt{c}}{2}r\right) + \frac{-4n^2 + 4n - 2}{n} \right\} \neq 0.$$

In this context, it is natural to pose the following problem:

Problem 2. Classify submanifolds  $(M, f \circ \iota_M)$  given by (1.2) having parallel mean curvature vector with respect to the normal connection in the sphere  $S^{n(n+2)-1}((n+1)c/(2n))$ .

The following proposition plays as a key in this section.

**Proposition 2.** Let  $M^{2n-1}$  be a real hypersurface of  $\mathbb{C}P^n(c)$  through an isometric immersion  $\iota_M$  and  $f : \mathbb{C}P^n(c) \to S^{n(n+2)-1}((n+1)c/(2n))$  the first standard minimal embedding. Then M is locally congruent to the geodesic sphere G(r) with  $\tan^2(\sqrt{c} r/2) = 2n + 1$  in  $\mathbb{C}P^n(c)$  if and only if the immersion  $f \circ \iota_M : M \to S^{n(n+2)-1}((n+1)c/(2n))$  has parallel mean curvature vector with respect to the normal connection. Moreover, this submanifold  $(M, f \circ \iota_M)$  is homogeneous in this ambient sphere.

Remark 2. The geodesic sphere G(r) in Proposition 2 is a Berger sphere, since  $\tan^2(\sqrt{c} r/2) = 2n + 1 > 2$ .

We next study the almost contact structure of our geodesic sphere in Propsition 2. For this purpose we review fundamental notions in contact geometry. Let  $M^{2m+1}$  be an almost contact metric manifold endowed with an almost contact metric structure  $(\phi, \xi, \eta, \langle , \rangle)$ . That is, this structure satisfies the following identities:

 $\phi^2 X = -X + \eta(X)\xi, \ \phi\xi = 0, \ \eta(\xi) = 1, \ \langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y)$ 

for all vectors X, Y on M. M is called a *Sasakian manifold* if the structure tensor  $\phi$  of M satisfies the following differential equation:

(4.2) 
$$(\nabla_X \phi)Y = \langle X, Y \rangle \xi - \eta(Y)X \text{ for } \forall X, Y \in TM,$$

where  $\nabla$  denotes the Riemannian connection of the Riemannian metric  $\langle , \rangle$  of M. A Sasakian manifold is called a *Sasakian space form* of constant  $\phi$ -sectional curvature c if the sectional curvature  $K(u, \phi u) := \langle R(u, \phi u)\phi u, u \rangle = c$  holds for every unit vector u orthogonal to  $\xi$ , where R is its curvature tensor. For construction of Sasakian space forms, see pp. 99-100 in [5].

In the following, we shall consider case that a real hypersurface M of  $\mathbb{C}P^n(c)$ is a Sasakian manifold with respect to the alomost contact metric structure  $(\phi, \xi, \eta, \langle , \rangle)$  induced from the Kähler structure J of the ambient space  $\mathbb{C}P^n(c)$ . Then it follows from (4.2) and (2.3) that  $\xi$  is principal. Hence, again by using (4.2) and (2.3) we find that Au = -u for each vector u orthogonal to  $\xi$ , so that our real hypersurface M is a member of totally  $\eta$ -umbilic hypersurfaces in  $\mathbb{C}P^n(c)$ . Hence, using the classification theorem of totally  $\eta$ -umbilic hypersurfaces in  $\mathbb{C}P^n(c)$ (see [13]), we see that the shape operator A of our Sasakian manifold M in  $\mathbb{C}P^n(c)$ is written as

(4.3) 
$$AX = -X + (c/4)\eta(X)\xi$$
 for each vector  $X \in TM$ .

Conversely, it follows from (2.3) and (4.3) that Equation (4.2) holds. Thus we know that M is a Sasakian manifold if and only if M has the shape operator A satisfying (4.3). Furthermore, M has constant  $\phi$ -sectional curvature c + 1.

Therefore, from the discussion here and Proposition 2 we obtain the following:

**Proposition 3** ([2]). The geodesic sphere G(r) with  $\tan^2(\sqrt{c} r/2) = 2n + 1$  in  $\mathbb{C}P^n(c)$  is a Sasakian manifold with respect to the almost contact metric structure induced from the ambient space  $\mathbb{C}P^n(c)$  if and only if c = 8n + 4. Moreover, this geodesic sphere is a Sasakian space form of constant  $\phi$ -sectional curvature c + 1.

By virtue of our discussion we establish the following:

**Theorem 6** ([11]). For each of c > 0,  $n(\geq 2)$  and N > n(n+2) - 1, there exists a (2n-1)-dimensional Riemannian submanifold  $M^{2n-1}$  in an N-dimensional sphere  $S^{N}(\tilde{c})$  of constant sectional curvature  $\tilde{c}$  with  $(n+1)c/(2n) \geq \tilde{c}$  satisfying the following three conditions:

- (1)  $M^{2n-1}$  is a homogeneous submanifold with nonzero parallel mean curvature vector with respect to the normal connection in the ambient sphere  $S^N(\tilde{c})$ ;
- (2)  $M^{2n-1}$  is a Bereger sphere;
- (3)  $M^{2n-1}$  has an almost contact metric structure  $(\phi, \xi, \eta, \langle , \rangle)$ . In particular, when c = 8n + 4, this submanifold M is a Sasakian space form of constant  $\phi$ -sectional curvature 8n + 5.

Moreover, for each of c > 0 and  $n \geq 2$ , when N = n(n+2)-1, there exists a (2n-1)-dimensional Riemannian submanifold  $M^{2n-1}$  in an N-dimensional sphere  $S^{N}(\tilde{c})$  of constant sectional curvature  $\tilde{c} = (n+1)c/(2n)$  satisfying the above conditions (1), (2), (3).

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