Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
42 (2009), pp. 17–57

DYNAMICS OF RATIONAL FUNCTIONS WITH SIEGEL DISKS AND POLYNOMIAL SEMIGROUPS

KOH KATAGATA

Communicated by Toshihiro Nakanishi

(Received: November 13, 2008)

Abstract. This article is based on the author's thesis, "Dynamics of rational functions and rational semigroups on the Riemann sphere", and consists of two topics. The first topic is about Blaschke products and rational functions with Siegel disks. The second one is about Julia sets of quartic polynomials and polynomial semigroups. Throughout this article, our interest is on topological and geometrical properties of Julia sets.

1. Introduction

Let f be a rational function on the Riemann sphere. In the theory of complex dynamics, there are two important sets called the Fatou set and the Julia set. The Fatou set is the set of normality in the sense of Montel for the family $\{f^n\}_{n=1}^{\infty}$, where $f^n = f \circ \cdots \circ f$ is the n-th iteration of f. The Julia set is the complement of the Fatou set. The dynamical behavior of f is stable on the Fatou set and "chaotic" on the Julia set, and recent advances in computer graphics have enabled us to notice that Julia sets have "fractal structure". In general, Julia sets are complicated and have abundant topological and geometrical properties. Moreover although Julia sets of polynomials are compact subsets of the complex plane with empty interior, there exists a quadratic polynomial such that its Julia set has positive Lebesgue measure. Therefore Julia sets have enough fascinations to warrant further study.

This article consists of two topics. Section 2 deals with Blaschke products and rational functions with Siegel disks. A Siegel disk of some polynomial with bounded type rotation number has the quasicircle boundary containing its critical point. For complex numbers λ and μ with $\lambda\mu \neq 1$ and a positive integer m, we consider two rational functions

$$E_{\lambda,\mu,m}(z) = z\left(\frac{z^m + \lambda}{\mu z^m + 1}\right)$$

2000 *Mathematics Subject Classification.* 37F10, 37F50, 37F99.
Key words and phrases. Complex dynamics, Julia sets, Siegel disks, Polynomial semigroups.
A summary of doctoral thesis at Shimane University.

and

$$F_{\lambda,\,\mu,\,m}(z) = z\left(\frac{z+\lambda}{\mu z+1}\right)^m.$$

The two rational functions $E_{\lambda,\,\mu,\,m}$ and $F_{\lambda,\,\mu,\,m}$ are semiconjugate via $S_m(z) = z^m$, namely

$$F_{\lambda,\,\mu,\,m} \circ S_m = S_m \circ E_{\lambda,\,\mu,\,m}.$$

We show that the Siegel disk of $E_{\lambda,\,\mu,\,m}$ centered at the origin with bounded type rotation number has the quasicircle boundary containing its critical point. This property is inherited by $F_{\lambda,\,\mu,\,m}$ because of the semiconjugate relationship. In order to show the above statement, we consider two Blaschke products

$$A_m(z) = e^{2\pi i\theta} z \left(\frac{z^m - a}{1-\bar{a}z^m}\right)\left(\frac{z^m - b}{1-\bar{b}z^m}\right)$$

and

$$B_m(z) = e^{2\pi i m\theta} z \left(\frac{z - a}{1-\bar{a}z}\right)^m\left(\frac{z - b}{1-\bar{b}z}\right)^m,$$

which are semiconjugate via S_m, and employ the quasiconformal surgery.

Section 3 deals with quartic polynomials and polynomial semigroups. It is known that if all finite critical points of a polynomial of degree greater than one belong to the attracting basin of the point at infinity, then the Julia set is totally disconnected and the polynomial restricted to the Julia set is topologically conjugate to the shift map on the symbol space. In the case that the Julia set of a polynomial of degree greater than one is neither connected nor totally disconnected, simplifying the dynamics of the polynomial on the Julia set into the dynamics of the shift map becomes a problem. In the case that the Julia set of a quartic polynomial is under the above assumption, we show that there exists a homeomorphism between the set of all components of the filled-in Julia set with the Hausdorff metric and some subset of the corresponding symbol space with the ordinary metric. Furthermore the quartic polynomial is topologically conjugate to the shift map via the homeomorphism. The result associates the Julia set of the quartic polynomial with that of a certain polynomial semigroup, namely there exists a homeomorphism between the Julia set of the quartic polynomial and that of the polynomial semigroup.

2. Blaschke Products and Rational Functions with Siegel Disks

In this section we study geometrical properties of the boundary of Siegel disks. A Siegel disk of some polynomial with bounded type rotation number has the quasicircle boundary containing its critical point. In order to construct such a Siegel disk not of a polynomial but of a rational function, we consider some Blaschke product and employ the quasiconformal surgery.

2.1. **Siegel disks of bounded type.**

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ with a fixed point of multiplier $e^{2\pi i\alpha}$ at the origin, where $\alpha \in [0,1]$ is irrational. Bryuno showed that if α is a Bryuno number, then f is linearizable near the origin. Yoccoz showed that if α is not a Bryuno number, then the quadratic polynomial $P_\alpha(z) = z^2 + e^{2\pi i\alpha}z$ is not linearizable near the origin, namely P_α is linearizable near the origin if and only if α is a Bryuno number. Moreover the following theorem holds if α is of bounded type.

Theorem 2.1 (Ghys-Douady-Herman-Shishikura-Świątek)**.** *If $\alpha \in [0,1]$ is irrational of bounded type, then the boundary of the Siegel disk Δ of P_α centered at the origin is a quasicircle containing its critical point $-e^{2\pi i\alpha}/2$.*

Moreover if $\alpha \in [0,1]$ is irrational of bounded type, then the following statements hold:

(a) (Petersen). The Julia set of P_α is locally connected and has measure zero.
(b) (McMullen). The Hausdorff dimension of the Julia set of P_α is less than two.
(c) (Graczyk-Jones). The Hausdorff dimension of $\partial\Delta$ is greater than one.

Conversely Petersen showed that if $\partial\Delta$ is a quasicircle containing the finite critical point $-e^{2\pi i\alpha}/2$ of P_α, then $\alpha \in [0,1]$ is of bounded type. Zakeri extended Theorem 2.1 to the case of cubic polynomials.

Theorem 2.2 (Zakeri, [44])**.** *Let f be a cubic polynomial with a fixed point of multiplier $e^{2\pi i\alpha}$ at the origin. If an irrational number $\alpha \in [0,1]$ is of bounded type, then the boundary of the Siegel disk of f centered at the origin is a quasicircle containing one or both critical points.*

Geyer showed the following theorem which extended Theorem 2.1 to the case of the polynomial $Q_{\alpha,m}(z) = e^{2\pi i\alpha}z(1 + z/m)^m$. The quadratic polynomial P_α is conformally conjugate to $Q_{\alpha,1}$.

Theorem 2.3 (Geyer, [19])**.** *Let m be a positive integer. If an irrational number $\alpha \in [0,1]$ is of bounded type, then the boundary of the Siegel disk of $Q_{\alpha,m}$ centered at the origin is a quasicircle containing its critical point $-m/(m+1)$.*

Let $F_{\lambda,\mu}(z) = z(z+\lambda)/(\mu z+1)$ with $\lambda\mu \neq 1$. The origin and the point at infinity are fixed points of $F_{\lambda,\mu}$ of multiplier λ and μ respectively. In the case that $\mu = 0$, $F_{\lambda,0}(z) = \lambda z + z^2$. Therefore the quadratic rational function $F_{\lambda,\mu}$ is considered as a perturbation of the quadratic polynomial $z \mapsto \lambda z + z^2$. In the case that $\lambda = e^{2\pi i\alpha}$ and α is irrational of bounded type, the author showed the following theorem which is a generalization of Theorem 2.1.

Theorem 2.4 ([24])**.** *If an irrational number $\alpha \in [0,1]$ is of bounded type, $\lambda = e^{2\pi i\alpha}$ and $\mu \in \overline{\mathbb{D}}$ with $\lambda\mu \neq 1$, then the boundary of the Siegel disk of $F_{\lambda,\mu}$ centered at the origin is a quasicircle containing its critical point.*

For complex numbers λ and μ with $\lambda\mu \neq 1$ and a positive integer m, we consider two rational functions

$$E_{\lambda,\mu,m}(z) = z\left(\frac{z^m + \lambda}{\mu z^m + 1}\right)$$

and

$$F_{\lambda,\mu,m}(z) = z\left(\frac{z+\lambda}{\mu z+1}\right)^m.$$

The two rational functions $E_{\lambda,\mu,m}$ and $F_{\lambda,\mu,m}$ are semiconjugate via $S_m(z) = z^m$, namely

$$F_{\lambda,\mu,m} \circ S_m = S_m \circ E_{\lambda,\mu,m}.$$

It is clear that $E_{\lambda,\mu,1} = F_{\lambda,\mu,1} = F_{\lambda,\mu}$. The origin is a fixed point of both $E_{\lambda,\mu,m}$ and $F_{\lambda,\mu,m}$ of multiplier λ and λ^m respectively, and the point at infinity is a fixed point of both $E_{\lambda,\mu,m}$ and $F_{\lambda,\mu,m}$ of multiplier μ and μ^m respectively. In the case that $\mu = 0$,

$$F_{\lambda,0,m}(z) = z\,(z+\lambda)^m.$$

Therefore the rational function $F_{\lambda,\mu,m}$ is considered as a perturbation of the polynomial $F_{\lambda,0,m}$. It is clear that $F_{\lambda,0,m}$ is conformally conjugate to $Q_{\alpha,m}$ if $\lambda^m = e^{2\pi i\alpha}$. In this section we show the following theorems which contain Theorem 2.4.

Theorem 2.5. *Let m be a positive integer and let $\mu \in \overline{\mathbb{D}}$. If an irrational number $\alpha \in [0,1]$ is of bounded type and $e^{2\pi i\alpha}\mu \neq 1$, then the boundary of the Siegel disk of $E_{\lambda,\mu,m}$ centered at the origin is a quasicircle containing its critical point, where $\lambda = e^{2\pi i\alpha}$.*

Theorem 2.6. *Let m be a positive integer and let $\mu \in \overline{\mathbb{D}}$. If an irrational number $\alpha \in [0,1]$ is of bounded type and $e^{2\pi i\alpha}\mu^m \neq 1$, then the boundary of the Siegel disk of $F_{\lambda,\mu,m}$ centered at the origin is a quasicircle containing its critical point, where λ satisfies that $\lambda^m = e^{2\pi i\alpha}$.*

Theorem 2.6	
$\mu = 0,\ m = 1$	Theorem 2.1
$\mu = 0$	Theorem 2.3
$m = 1$	Theorem 2.4

Table 1. Special cases of Theorem 2.6

Theorem 2.6 contains Theorems 2.1, 2.3 and 2.4. Moreover we obtain the following two corollaries.

Corollary 2.7. *Let m be a positive integer. If α and β in $[0,1]$ are irrational of bounded type and $e^{2\pi i(\alpha+\beta)} \neq 1$, then the boundary of the Siegel disk of $E_{\lambda,\mu,m}$ centered at the origin and that of the Siegel disk of $E_{\lambda,\mu,m}$ centered at the point at infinity are quasicircles containing its critical point, where $\lambda = e^{2\pi i\alpha}$ and $\mu = e^{2\pi i\beta}$.*

Corollary 2.8. *Let m be a positive integer. If α and β in $[0,1]$ are irrational of bounded type and $e^{2\pi i(\alpha+\beta)} \neq 1$, then the boundary of the Siegel disk of $F_{\lambda,\mu,m}$ centered at the origin and that of the Siegel disk of $F_{\lambda,\mu,m}$ centered at the point at infinity are quasicircles containing its critical point, where λ and μ satisfy that $\lambda^m = e^{2\pi i\alpha}$ and $\mu^m = e^{2\pi i\beta}$.*

2.2. **Blaschke product models.**

Existence of Blaschke product models.
Let m be a positive integer. We consider the Blaschke product

$$B(z) = e^{2\pi i m\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)^m \left(\frac{z-b}{1-\bar{b}z}\right)^m$$

of degree $2m+1$ with $a\bar{b} \neq 1$ and $0 < |a| \leq |b| < \infty$. Let $\lambda = abe^{2\pi i\theta}$ and let $\mu = \bar{a}\bar{b}e^{-2\pi i\theta}$. The derivative B' of B is

$$B'(z) = \frac{e^{2\pi i m\theta}}{(1-\bar{a}z)^2(1-\bar{b}z)^2} \left(\frac{z-a}{1-\bar{a}z}\right)^{m-1} \left(\frac{z-b}{1-\bar{b}z}\right)^{m-1} g(z),$$

where

$$\begin{aligned} g(z) = \bar{a}\bar{b}z^4 &+ \Big\{-(m+1)(\bar{a}+\bar{b}) + (m-1)\bar{a}\bar{b}(a+b)\Big\}z^3 \\ &+ \Big\{2m+1-(2m-1)|ab|^2 + |a+b|^2\Big\}z^2 \\ &+ \Big\{-(m+1)(a+b) + (m-1)ab(\bar{a}+\bar{b})\Big\}z + ab. \end{aligned}$$

Then multipliers of fixed points $z=0$ and $z=\infty$ are λ^m and μ^m respectively. Let c_1, c_2, $c_3 = 1/\bar{c}_2$ and $c_4 = 1/\bar{c}_1$ be the solutions of the equation $g(z)=0$. Therefore critical points of B are a, $1/\bar{a}$, b, $1/\bar{b}$ c_1, c_2, c_3 and c_4, and multiplicities of critical points a, $1/\bar{a}$, b and $1/\bar{b}$ are $m-1$. Since c_1, c_2, c_3 and c_4 are the solutions of the equation $g(z)=0$, we obtain that

$$\begin{aligned} g(z) &= \bar{a}\bar{b}(z-c_1)(z-c_2)(z-c_3)(z-c_4) \\ &= \bar{a}\bar{b}\Big\{z^4 - C_3z^3 + C_2z^2 - C_1z + C_0\Big\}, \end{aligned}$$

where

$$C_0 = \frac{c_1c_2}{\bar{c}_1\bar{c}_2}, \quad C_1 = \frac{c_1}{\bar{c}_1}\left(c_2 + \frac{1}{\bar{c}_2}\right) + \frac{c_2}{\bar{c}_2}\left(c_1 + \frac{1}{\bar{c}_1}\right),$$

$$C_2 = \frac{c_1}{\bar{c}_1} + \frac{c_2}{\bar{c}_2} + \left(c_1 + \frac{1}{\bar{c}_1}\right)\left(c_2 + \frac{1}{\bar{c}_2}\right), \quad C_3 = c_1 + \frac{1}{\bar{c}_1} + c_2 + \frac{1}{\bar{c}_2}.$$

Comparing coefficients of two representations of $g(z)$ implies that

$$c_1 + \frac{1}{\bar{c}_1} + c_2 + \frac{1}{\bar{c}_2} = \frac{(m+1)(\bar{a}+\bar{b}) - (m-1)(a+b)\bar{a}\bar{b}}{\bar{a}\bar{b}}, \tag{1}$$

$$\frac{c_1}{\bar{c}_1} + \frac{c_2}{\bar{c}_2} + \left(c_1 + \frac{1}{\bar{c}_1}\right)\left(c_2 + \frac{1}{\bar{c}_2}\right) = \frac{2m+1-(2m-1)|ab|^2 + |a+b|^2}{\bar{a}\bar{b}}, \tag{2}$$

$$\frac{c_1}{\bar{c}_1}\left(c_2 + \frac{1}{\bar{c}_2}\right) + \frac{c_2}{\bar{c}_2}\left(c_1 + \frac{1}{\bar{c}_1}\right) = \frac{(m+1)(a+b) - (m-1)(\bar{a}+\bar{b})ab}{\bar{a}\bar{b}}, \tag{3}$$

$$\frac{c_1 c_2}{\bar{c}_1 \bar{c}_2} = \frac{ab}{\bar{a}\bar{b}}. \tag{4}$$

Eliminating c_1 and $\bar{c}_1$ from the equations (1), (2) and (4) gives that

$$\begin{aligned} &|a+b|^2 - (m+1)\left(c_2 + \frac{1}{\bar{c}_2}\right)(\bar{a}+\bar{b}) - \left(\frac{\bar{c}_2}{c_2}\right)ab \\ &\quad + \left\{\left(c_2 + \frac{1}{\bar{c}_2}\right)^2 - \frac{c_2}{\bar{c}_2}\right\}\bar{a}\bar{b} + (m-1)\left(c_2 + \frac{1}{\bar{c}_2}\right)(a+b)\bar{a}\bar{b} \\ &\qquad + 2m+1-(2m-1)|ab|^2 = 0, \end{aligned} \tag{5}$$

and eliminating c_1 and $\bar{c}_1$ from the equations (1), (3) and (4) gives that

$$\begin{aligned} &\frac{\bar{c}_2}{c_2}\left(c_2 + \frac{1}{\bar{c}_2}\right)ab + (m+1)\left(\frac{c_2}{\bar{c}_2}\right)(\bar{a}+\bar{b}) - (m-1)\left(\frac{c_2}{\bar{c}_2}\right)(a+b)\bar{a}\bar{b} \\ &\quad = \frac{c_2}{\bar{c}_2}\left(c_2 + \frac{1}{\bar{c}_2}\right)\bar{a}\bar{b} + (m+1)(a+b) - (m-1)(\bar{a}+\bar{b})ab. \end{aligned} \tag{6}$$

We obtain that

$$\begin{aligned} &|a+b|^2 - 2(m+1)e^{2\pi i\varphi}(\bar{a}+\bar{b}) - e^{2\pi i(-2\varphi)}ab + 3e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b} \\ &\quad + 2(m-1)e^{2\pi i\varphi}(a+b)\bar{a}\bar{b} + 2m+1-(2m-1)|ab|^2 = 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} &e^{2\pi i(-2\varphi)}ab + \frac{m+1}{2}e^{2\pi i\varphi}(\bar{a}+\bar{b}) - \frac{m-1}{2}e^{2\pi i\varphi}(a+b)\bar{a}\bar{b} \\ &\quad = e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b} + \frac{m+1}{2}e^{2\pi i(-\varphi)}(a+b) - \frac{m-1}{2}e^{2\pi i(-\varphi)}(\bar{a}+\bar{b})ab \end{aligned} \tag{8}$$

by substituting $c_2 = e^{2\pi i\varphi}$ into the equations (5) and (6). Eliminating ab from the equations (7) and (8) gives that

$$
\begin{aligned}
(9)\quad & |a+b|^2 - \frac{3}{2}(m+1)e^{2\pi i\varphi}(\bar{a}+\bar{b}) \\
& - \frac{m+1}{2}e^{2\pi i(-\varphi)}(a+b) + 2e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b} + \frac{m-1}{2}e^{2\pi i(-\varphi)}(\bar{a}+\bar{b})ab \\
& + \frac{3}{2}(m-1)e^{2\pi i\varphi}(a+b)\bar{a}\bar{b} + 2m+1-(2m-1)|ab|^2 = 0.
\end{aligned}
$$

Let $\zeta = a+b$ and then

$$
\begin{aligned}
(10)\quad & |\zeta|^2 - \frac{3}{2}(m+1)e^{2\pi i\varphi}\bar{\zeta} \\
& - \frac{m+1}{2}e^{2\pi i(-\varphi)}\zeta + 2e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b} + \frac{m-1}{2}e^{2\pi i(-\varphi)}ab\bar{\zeta} \\
& + \frac{3}{2}(m-1)e^{2\pi i\varphi}\bar{a}\bar{b}\zeta + 2m+1-(2m-1)|ab|^2 = 0.
\end{aligned}
$$

The real part of the left side of the equation (10) is

$$
\begin{aligned}
(11)\quad & x^2+y^2-2x\Big\{(m+1)\cos 2\pi\varphi - (m-1)r\cos 2\pi(\varphi+\theta+\omega)\Big\} \\
& - 2y\Big\{(m+1)\sin 2\pi\varphi + (m-1)r\sin 2\pi(\varphi+\theta+\omega)\Big\} \\
& + 2r\cos 2\pi(2\varphi+\theta+\omega) + 2m+1-(2m-1)r^2 = 0,
\end{aligned}
$$

and the imaginary part of the left side of the equation (10) is

$$
\begin{aligned}
(12)\quad & y\Big\{(m+1)\cos 2\pi\varphi + (m-1)r\cos 2\pi(\varphi+\theta+\omega)\Big\} \\
& - x\Big\{(m+1)\sin 2\pi\varphi - (m-1)r\sin 2\pi(\varphi+\theta+\omega)\Big\} \\
& + 2r\sin 2\pi(2\varphi+\theta+\omega) = 0,
\end{aligned}
$$

where $\zeta = x+iy$ and $\mu = \bar{a}\bar{b}e^{-2\pi i\theta} = re^{2\pi i\omega}$. The solutions of simultaneous equations (11) and (12) are

$$
\begin{aligned}
x = & \Big\{(m+1)^2+(m-1)^2r^2+2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-1} \\
& \times\Big\{C_4\cos 2\pi\varphi + C_5\cos 2\pi(\varphi+\theta+\omega) \\
& + C_6\cos 2\pi(3\varphi+\theta+\omega) + C_7\cos 2\pi(3\varphi+2\theta+2\omega)\Big\}
\end{aligned}
$$

and

$$y = \Big\{(m+1)^2 + (m-1)^2r^2 + 2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-1}$$
$$\times \Big\{C_4 \sin 2\pi\varphi - C_5 \sin 2\pi(\varphi+\theta+\omega)$$
$$+C_6 \sin 2\pi(3\varphi+\theta+\omega) - C_7 \sin 2\pi(3\varphi+2\theta+2\omega)\Big\},$$

where

$$C_4 = (m+1)^2(2m+1) - 2m(m^2-1)r^2, \qquad C_6 = -(m+1)^2 r,$$
$$C_5 = 2m(m^2-1)r - (m-1)^2(2m-1)r^3, \quad C_7 = -(m-1)^2r^2.$$

Hence $\zeta = x + iy$ satisfies the equation (10). Conversely we show the following theorem.

Theorem 2.9. *Let $\mu = re^{2\pi i\omega} \in \overline{\mathbb{D}}$ and let $a = a(\theta,\varphi)$ and $b = b(\theta,\varphi)$ with $|a| \le |b|$ be complex numbers satisfying relations $a+b = x+iy$ and $ab = re^{-2\pi i(\theta+\omega)}$, namely a and b are the solutions of the equation*

$$Z^2 - (x+iy)Z + re^{-2\pi i(\theta+\omega)} = 0, \tag{†}$$

where x and y are as above and $(\theta,\varphi) \in [0,1]^2$. Then the following holds:

(a) *If $r = 0$, then $a = 0$ and $b = (2m+1)e^{2\pi i\varphi}$.*
(b) *If $0 < r < 1$, then $0 < |a| < 1 < |b| < \infty$.*
(c) *If $r = 1$ and $2\varphi+\theta+\omega \equiv 0 \pmod 1$, then $a = b = e^{2\pi i\varphi}$.*
(d) *If $r = 1$ and $2\varphi+\theta+\omega \not\equiv 0 \pmod 1$, then $0 < |a| < 1 < |b| < \infty$.*
(e) *In the case* (a), (b) *or* (d),

$$B(z) = B_{\theta,\varphi,m}(z) = e^{2\pi i m\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)^m \left(\frac{z-b}{1-\bar{b}z}\right)^m$$

is a Blaschke product of degree $2m+1$ and the point at infinity is a fixed point of B with multiplier μ^m. Moreover $z = e^{2\pi i\varphi}$ is a critical point of B and $B|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is a homeomorphism, where $\mathbb{T}$ is the unit circle.

Proof. First, we show the following lemma.

Lemma 2.10. *The inequality $|x+iy| \ge 2$ holds. Moreover the equality holds if and only if $r = 1$ and $2\varphi+\theta+\omega \equiv 0 \pmod 1$ hold.*

Proof of Lemma 2.10. Since

$$\cos 2\pi\cdot 2(2\varphi+\theta+\omega) = 2\cos^2 2\pi(2\varphi+\theta+\omega) - 1$$

and

$$\cos 2\pi\cdot 3(2\varphi+\theta+\omega) = 4\cos^3 2\pi(2\varphi+\theta+\omega) - 3\cos 2\pi(2\varphi+\theta+\omega),$$

we obtain that

$$
\begin{aligned}
|x+iy|^2 &= \Big\{(m+1)^2+(m-1)^2r^2+2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad \times \left|C_4e^{2\pi i\varphi}+C_5e^{-2\pi i(\varphi+\theta+\omega)}+C_6e^{2\pi i(3\varphi+\theta+\omega)}+C_7e^{-2\pi i(3\varphi+2\theta+2\omega)}\right|^2 \\
&= \Big\{(m+1)^2+(m-1)^2r^2+2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad \times \Bigg\{C_4^2+C_5^2+C_6^2+C_7^2-2C_4C_7-2C_5C_6 \\
&\quad +2\Big(C_4C_5+C_4C_6+C_5C_7-3C_6C_7\Big)\cos 2\pi(2\varphi+\theta+\omega) \\
&\quad +4\Big(C_4C_7+C_5C_6\Big)\cos^2 2\pi(2\varphi+\theta+\omega)+8C_6C_7\cos^3 2\pi(2\varphi+\theta+\omega)\Bigg\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|x+iy|^2 &= \Big\{(m+1)^2+(m-1)^2r^2+2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad \times \Bigg[4m^6+20m^5+41m^4+44m^3+26m^2+8m+1 \\
&\quad +\left(-4m^6-12m^5-5m^4+12m^3+14m^2+8m+3\right)r^2 \\
&\quad +\left(-4m^6+12m^5-5m^4-12m^3+14m^2-8m+3\right)r^4 \\
&\quad +\left(4m^6-20m^5+41m^4-44m^3+26m^2-8m+1\right)r^6 \\
&\quad +\Big\{\left(8m^6+16m^5-10m^4-48m^3-44m^2-16m-2\right)r \\
&\quad +\left(-16m^6+44m^4-24m^2-4\right)r^3 \\
&\quad +\left(8m^6-16m^5-10m^4+48m^3-44m^2+16m-2\right)r^5\Big\}\cos 2\pi(2\varphi+\theta+\omega) \\
&\quad +\Big\{\left(-16m^5-20m^4+16m^3+24m^2-4\right)r^2 \\
&\quad +\left(16m^5-20m^4-16m^3+24m^2-4\right)r^4\Big\}\cos^2 2\pi(2\varphi+\theta+\omega) \\
&\quad +\left(8m^4-16m^2+8\right)r^3\cos^3 2\pi(2\varphi+\theta+\omega)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \Big\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad \times \Bigg[\Big[(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega)\Big] \\
&\quad \times \Big[(m+1)^2(2m+1)^2 - 2(4m^4-5m^2-1)r^2 + (m-1)^2(2m-1)^2 r^4 \\
&\quad + \Big\{-4m(m+1)(2m+1)r + 4m(m-1)(2m-1)r^3\Big\} \cos 2\pi(2\varphi+\theta+\omega) \\
&\quad + 4(m^2-1)r^2 \cos^2 2\pi(2\varphi+\theta+\omega)\Big]\Bigg] \\
&= \Big\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-1} \\
&\quad \times \Bigg[(m+1)^2(2m+1)^2 - 2(4m^4-5m^2-1)r^2 + (m-1)^2(2m-1)^2 r^4 \\
&\quad + 4mr\Big\{-(m+1)(2m+1) + (m-1)(2m-1)r^2\Big\} \cos 2\pi(2\varphi+\theta+\omega) \\
&\quad + 4(m^2-1)r^2 \cos^2 2\pi(2\varphi+\theta+\omega)\Bigg].
\end{aligned}
$$

Let $X = \cos 2\pi(2\varphi+\theta+\omega)$ and we consider the function

$$
\begin{aligned}
f(X) &= \Big\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r X\Big\}^{-1} \\
&\quad \times \Bigg[(m+1)^2(2m+1)^2 - 2(4m^4-5m^2-1)r^2 + (m-1)^2(2m-1)^2 r^4 \\
&\quad + 4mr\Big\{-(m+1)(2m+1) + (m-1)(2m-1)r^2\Big\} X + 4(m^2-1)r^2 X^2\Bigg].
\end{aligned}
$$

Then the function f is monotone decreasing on $[-1, 1]$ and

$$f(1) = \Big\{2m+1-(2m-1)r\Big\}^2.$$

In the case that $0 \le r < 1$, we obtain that

$$|x+iy| \ge \sqrt{f(1)} = 2m+1-(2m-1)r > 2.$$

In the case that $r = 1$ and $2\varphi+\theta+\omega \not\equiv 0 \pmod 1$, we obtain that

$$|x+iy| > \sqrt{f(1)} = 2m+1-(2m-1)\cdot 1 = 2.$$

Moreover in the case that $r = 1$ and $2\varphi+\theta+\omega \equiv 0 \pmod 1$, we obtain that

$$|x+iy| = \sqrt{f(1)} = 2m+1-(2m-1)\cdot 1 = 2.$$

$\square$

Proof of (a). It is clear.

Proof of (b). By Lemma 2.10, $|a+b| = |x+iy| > 2$. In the case that $0 < r < 1$, either $0 < |a| < 1 \leq |b| < \infty$ or $0 < |a| \leq |b| \leq 1$ hold since $|a||b| = r$. If $0 < |a| \leq |b| \leq 1$, then

$$2 < |a+b| \leq |a| + |b| \leq 2.$$

This is a contradiction and hence the situation $0 < |a| < 1 \leq |b| < \infty$ happens. If $|b| = 1$, then

$$2 < |a+b| \leq |a| + |b| = |a| + 1 < 2.$$

This is a contradiction. Therefore the equation (†) does not have double roots and $0 < |a| < 1 < |b| < \infty$.

Proof of (c). By assumptions, we obtain that $x + iy = 2e^{2\pi i\varphi}$ and $re^{-2\pi i(\theta+\omega)} = e^{2\pi i \cdot 2\varphi}$. Therefore the equation (†) is

$$Z^2 - 2e^{2\pi i\varphi} Z + e^{2\pi i \cdot 2\varphi} = 0$$

and hence $a = b = e^{2\pi i\varphi}$.

Proof of (d). By Lemma 2.10, $|a+b| = |x+iy| > 2$. In the case that $r = 1$, either $0 < |a| < 1 < |b| < \infty$ or $|a| = |b| = 1$ hold since $|a||b| = 1$. If $|a| = |b| = 1$, then

$$2 < |a+b| \leq |a| + |b| = 2.$$

This is a contradiction. Therefore the equation (†) does not have double roots and $0 < |a| < 1 < |b| < \infty$.

Proof of (e). Let

$$u(z) = \left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{z-b}{1-\bar{b}z}\right) = \frac{z^2 - (a+b)z + ab}{\bar{a}\bar{b}z^2 - (\bar{a}+\bar{b})z + 1}.$$

The necessary and sufficient condition that the degree of the Blaschke product B be $2m+1$ is that the function u be not constant, and then the necessary and sufficient condition that the degree of the Blaschke product B be one is that the function u be constant. In the case that $r = 0$, the function u is not constant since

$$u(z) = \frac{z^2 - (2m+1)e^{2\pi i\varphi} z}{-(2m+1)e^{-2\pi i\varphi} z + 1}.$$

If $r \neq 0$, then

$$u(z) = \frac{1}{\bar{a}\bar{b}} \cdot \frac{\bar{a}\bar{b}z^2 - \bar{a}\bar{b}(a+b)z + |ab|^2}{\bar{a}\bar{b}z^2 - (\bar{a}+\bar{b})z + 1}.$$

In the case that $0 < r < 1$, the degree of the Blaschke product B is $2m+1$ since $|ab| = r < 1$. In the case that $r = 1$, we obtain that

$$\bar{a}\bar{b}(a+b) - \left(\bar{a}+\bar{b}\right) = \frac{-2me^{-2\pi i(3\varphi+\theta+\omega)}\left\{e^{2\pi i(2\varphi+\theta+\omega)} - 1\right\}^3}{m^2 + 1 + (m^2-1)\cos 2\pi(2\varphi+\theta+\omega)}.$$

Therefore in the case that $r = 1$ and $2\varphi + \theta + \omega \not\equiv 0 \pmod 1$, the degree of the Blaschke product B is $2m+1$. On the other hand, if $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$, then

$$u(z) = \frac{1}{\bar{a}\bar{b}} = e^{2\pi i \cdot 2\varphi}$$

and the degree of the Blaschke product B is one. It is clear that the point at infinity is a fixed point of B with multiplier μ^m. Moreover it is clear that $g(e^{2\pi i\varphi}) = 0$ and hence $z = e^{2\pi i\varphi}$ is a critical point of B, where

$$B'(z) = \frac{e^{2\pi i m\theta}}{(1-\bar{a}z)^2(1-\bar{b}z)^2}\left(\frac{z-a}{1-\bar{a}z}\right)^{m-1}\left(\frac{z-b}{1-\bar{b}z}\right)^{m-1} g(z)$$

and

$$\begin{aligned} g(z) = \bar{a}\bar{b}z^4 &+ \Big\{-(m+1)(\bar{a}+\bar{b}) + (m-1)\bar{a}\bar{b}(a+b)\Big\}z^3 \\ &\quad + \Big\{2m+1-(2m-1)|ab|^2 + |a+b|^2\Big\}z^2 \\ &\qquad + \Big\{-(m+1)(a+b) + (m-1)ab(\bar{a}+\bar{b})\Big\}z + ab. \end{aligned}$$

Finally we show that two critical points of B other than a, $1/\bar{a}$, b, $1/\bar{b}$ (if $m \geq 2$) and $e^{2\pi i\varphi}$ belong to $\hat{\mathbb{C}} \setminus \mathbb{T}$. In the case that $r = 0$, we obtain that

$$g(z) = -(m+1)(2m+1)e^{-2\pi i\varphi} z \left(z - e^{2\pi i\varphi}\right)^2.$$

Therefore critical points of B are b, $1/\bar{b}$ (if $m \geq 2$), 0, ∞ and $e^{2\pi i\varphi}$. In the case that $r \neq 0$, let

$$h(z) = z^2 + \frac{e^{2\pi i\varphi}}{C_{10}}\left\{C_9 e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} + C_8 e^{-2\pi i(2\varphi+\theta+\omega)} + C_9\right\} z + e^{-2\pi i\cdot 2(\varphi+\theta+\omega)},$$

where

$$\begin{aligned} C_8 &= -(m+1)^3(2m+1) + 2(2m^4 - m^2 - 1)r^2 - (m-1)^3(2m-1)r^4, \\ C_9 &= (m+1)^3 r - (m-1)^3 r^3, \\ C_{10} &= (m+1)^2 r + (m-1)^2 r^3 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega). \end{aligned}$$

Then we can factor $r^{-1}e^{-2\pi i(\theta+\omega)}g(z)$ as

$$\frac{1}{r}\cdot e^{-2\pi i(\theta+\omega)}\cdot g(z) = \left(z - e^{2\pi i\varphi}\right)^2 \cdot h(z).$$

Let

$$h_1(z) = \frac{e^{2\pi i\varphi}}{C_{10}}\left\{C_9 e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} + C_8 e^{-2\pi i(2\varphi+\theta+\omega)} + C_9\right\} z$$

and

$$h_2(z) = z^2 + e^{-2\pi i\cdot 2(\varphi+\theta+\omega)}.$$

For any z in $\mathbb{T}$, we obtain that $|h_2(z)| \leq 2$.

Lemma 2.11. *The inequality $|h_1(z)| > 2$ holds on $\mathbb{T}$.*

Proof of Lemma 2.11. In the case that $0 < r < 1$, we obtain that

$$\begin{aligned}|h_1(z)| &= \frac{1}{|C_{10}|}\left|C_9 e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} + C_8 e^{-2\pi i(2\varphi+\theta+\omega)} + C_9\right| \\ &\geq \frac{|C_8| - 2|C_9|}{|C_{10}|} = \frac{-C_8 - 2C_9}{|C_{10}|} \geq v(m,r)\end{aligned}$$

on $\mathbb{T}$, where

$$\begin{aligned}v(m,r) = &\left\{(3m-1)(m+1)r + (m-1)^2 r^3\right\}^{-1} \\ &\times \left\{(m+1)^3(2m+1) - 2(m+1)^3 r - 2(2m^4 - m^2 - 1)r^2\right. \\ &\left. + 2(m-1)^3 r^3 + (m-1)^3(2m-1)r^4\right\}.\end{aligned}$$

Since the function $r \mapsto v(m,r)$ is monotone decreasing on $(0,1]$ and $v(m,1) = 2$, we obtain that $|h_1(z)| > 2$ on $\mathbb{T}$. In the case that $r = 1$ and $2\varphi + \theta + \omega \not\equiv 0 \pmod 1$, we obtain that

$$\begin{aligned}|h_1(z)| &= \frac{|C_9|}{|C_{10}|}\left|e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} + \frac{C_8}{C_9} e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right| \\ &= \frac{C_9}{|C_{10}|}\left|\left\{e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right\}^2 + \left(\frac{C_8}{C_9} - 2\right) e^{-2\pi i(2\varphi+\theta+\omega)}\right| \\ &\geq \frac{C_9}{|C_{10}|}\left|\left|e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right|^2 - \left|\frac{C_8}{C_9} - 2\right|\right| \\ &= \frac{C_9}{|C_{10}|}\left|\left|e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right|^2 - \frac{4(4m^2+1)}{3m^2+1}\right| \\ &\geq \frac{3m^2+1}{2m^2}\left\{\frac{4(4m^2+1)}{3m^2+1} - \left|e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right|^2\right\} \\ &> \frac{3m^2+1}{2m^2}\left\{\frac{4(4m^2+1)}{3m^2+1} - 4\right\} \\ &= 2\end{aligned}$$

on $\mathbb{T}$. □

By the Rouché's theorem, the number of roots of $h(z) = h_1(z) + h_2(z)$ on $\mathbb{D}$ is one since $|h_1(z)| > 2 \geq |h_2(z)|$ on $\mathbb{T}$ and the number of roots of $h_1(z)$ on $\mathbb{D}$ is one. Hence one of critical points of B other than a, $1/\bar{a}$, b, $1/\bar{b}$ (if $m \geq 2$) and $e^{2\pi i\varphi}$ belongs to $\mathbb{D}$. Since critical points of a Blaschke product are symmetric with respect to the unit circle, the other one critical point of B belongs to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. In this case, the inverse image $B^{-1}(\mathbb{T})$ of $\mathbb{T}$ is the union of $\mathbb{T}$ and a figure eight $\mathsf{8}$ which crosses at $z = e^{2\pi i\varphi}$. See Figure 1. Then $B|_{\mathsf{8}} : \mathsf{8} \to \mathbb{T}$ is a $2m$-to-1 map and therefore $B|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is a homeomorphism. □

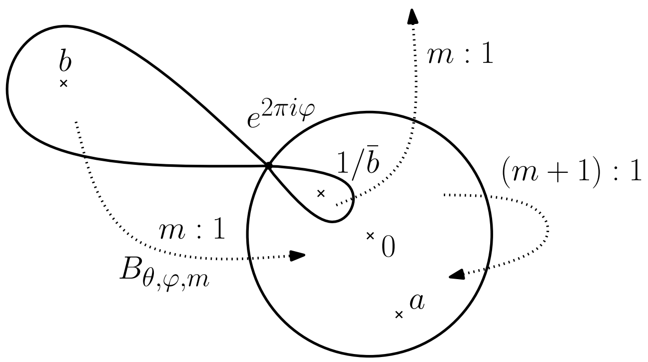

FIGURE 1. The inverse image $B^{-1}_{\theta,\varphi,m}(\mathbb{T})$ of the unit circle $\mathbb{T}$.

Remark 2.12. Two complex numbers $a = a(\theta,\varphi)$ and $b = b(\theta,\varphi)$ satisfy that

$$a(\theta+1,\varphi) = a(\theta,\varphi) = a(\theta,\varphi+1)$$

and

$$b(\theta+1,\varphi) = b(\theta,\varphi) = b(\theta,\varphi+1).$$

Rotation numbers of Blaschke products.
Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism and let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a lift of f via $x \mapsto e^{2\pi ix}$ which satisfies that $\tilde{f}(x+1) = \tilde{f}(x)+1$ for all $x \in \mathbb{R}$. The lift $\tilde{f}$ of f is unique up to addition of an integer constant. The rotation number $\rho(\tilde{f})$ of $\tilde{f}$ is defined as

$$\rho(\tilde{f}) = \lim_{n\to\infty} \frac{\tilde{f}^n(x)}{n},$$

which is independent of $x \in \mathbb{R}$. The rotation number $\rho(f)$ is defined as the residue class of $\rho(\tilde{f})$ modulo $\mathbb{Z}$. Poincaré showed that the rotation number is rational with denominator q if and only if f has a periodic point with period q. The following theorem is important (see [28]).

Theorem 2.13. *Let $\mathcal{F}$ be the set of all orientation-preserving homeomorphisms from the unit circle onto itself with the topology of uniform convergence. Then the rotation number function $\rho : \mathcal{F} \to \mathbb{R}/\mathbb{Z}$ defined as $f \mapsto \rho(f)$ is continuous.*

Let $a(\theta,\varphi)$ and $b(\theta,\varphi)$ be as in Theorem 2.9. We define a map $\Gamma_m : [0,1]^3 \to \mathbb{T}$ as

$$\Gamma_m(x,\theta,\varphi) = \left(\frac{e^{2\pi ix} - a(\theta,\varphi)}{1-\overline{a(\theta,\varphi)}\,e^{2\pi ix}}\right)^m \left(\frac{e^{2\pi ix} - b(\theta,\varphi)}{1-\overline{b(\theta,\varphi)}\,e^{2\pi ix}}\right)^m$$

and a map $H_m : [0,1]^4 \to \mathbb{T}$ as

$$H_m(x,\theta,\varphi,t) = \left(\frac{e^{2\pi ix} - a(\theta,\varphi,t)}{1-\overline{a(\theta,\varphi,t)}\,e^{2\pi ix}}\right)^m \left(\frac{e^{2\pi ix} - b(\theta,\varphi,t)}{1-\overline{b(\theta,\varphi,t)}\,e^{2\pi ix}}\right)^m,$$

where

$$a(\theta, \varphi, t) = (1-t)a(\theta, \varphi) + te^{2\pi i \varphi}$$

and

$$b(\theta, \varphi, t) = (1-t)b(\theta, \varphi) + te^{2\pi i \varphi}.$$

In the case that $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$, we obtain that $\Gamma_m(x, \theta, \varphi) = e^{2\pi i \cdot 2m\varphi}$. The following three lemmas play important roles in the proof of Theorem 2.17.

Lemma 2.14. *A map $H_m(\,\cdot\,, \theta, \varphi, \,\cdot\,) : [0,1]^2 \to \mathbb{T}$ is a homotopy between a loop $x \mapsto \Gamma_m(x, \theta, \varphi)$ and a constant loop $x \mapsto e^{2\pi i \cdot 2m\varphi}$ for any $(\theta, \varphi) \in [0,1]^2$.*

Proof. It is clear since $H_m(\,\cdot\,, \theta, \varphi, 0) = \Gamma_m(\,\cdot\,, \theta, \varphi)$ and $H_m(\,\cdot\,, \theta, \varphi, 1) = e^{2\pi i \cdot 2m\varphi}$. □

Lemma 2.15. *A map $H_m(x, \,\cdot\,, \varphi, \,\cdot\,) : [0,1]^2 \to \mathbb{T}$ is a homotopy between a loop $\theta \mapsto \Gamma_m(x, \theta, \varphi)$ and a constant loop $\theta \mapsto e^{2\pi i \cdot 2m\varphi}$ for any $(x, \varphi) \in [0,1]^2$.*

Proof. It is clear since $H_m(x, \,\cdot\,, \varphi, 0) = \Gamma_m(x, \,\cdot\,, \varphi)$ and $H_m(x, \,\cdot\,, \varphi, 1) = e^{2\pi i \cdot 2m\varphi}$. □

Lemma 2.16. *A map $H_m(x, \theta, \,\cdot\,, \,\cdot\,) : [0,1]^2 \to \mathbb{T}$ is a homotopy between a loop $\varphi \mapsto \Gamma_m(x, \theta, \varphi)$ and a loop $\varphi \mapsto e^{2\pi i \cdot 2m\varphi}$ for any $(x, \theta) \in [0,1]^2$.*

Proof. It is clear since $H_m(x, \theta, \,\cdot\,, 0) = \Gamma_m(x, \theta, \,\cdot\,)$ and $H_m(x, \theta, \,\cdot\,, 1) = e^{2\pi i \cdot 2m\varphi}$. □

Lemma 2.14 and Lemma 2.15 imply that

$$\arg(\Gamma_m(x+1, \theta, \varphi)) = \arg(\Gamma_m(x, \theta, \varphi)) = \arg(\Gamma_m(x, \theta+1, \varphi)),$$

and Lemma 2.16 implies that

$$\frac{1}{2\pi}\arg(\Gamma_m(x, \theta, \varphi+1)) = \frac{1}{2\pi}\arg(\Gamma_m(x, \theta, \varphi)) + 2m.$$

Theorem 2.17. *Let $\alpha \in [0,1]$ and let $\mu = re^{2\pi i\omega} \in \overline{\mathbb{D}}$. Besides let $a = a(\theta, \varphi)$ and $b = b(\theta, \varphi)$ be as in Theorem 2.9. Then for the Blaschke product*

$$B_{\theta, \varphi, m}(z) = e^{2\pi i m\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)^m \left(\frac{z-b}{1-\bar{b}z}\right)^m,$$

$B_{\theta, \varphi, m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism. Moreover

(a) *If $0 \le r < 1$, then there exists $(\theta_0, \varphi_0) \in [0,1]^2$ such that $\rho(B_{\theta_0, \varphi_0, m}|_{\mathbb{T}}) = \alpha$.*
(b) *If $r = 1$ and $\alpha + m\omega \not\equiv 0 \pmod 1$, then there exists $(\theta_0, \varphi_0) \in [0,1]^2$ such that $\rho(B_{\theta_0, \varphi_0, m}|_{\mathbb{T}}) = \alpha$ and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod 1$.*

Proof. In the case that $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$,

$$B_{\theta, \varphi, m}(z) = e^{2\pi i m(2\varphi+\theta)} z = e^{2\pi i(-m\omega)} z.$$

Therefore $B_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism and its rotation number satisfies that $\rho(B_{\theta,\varphi,m}|_{\mathbb{T}}) \equiv -m\omega \pmod 1$. In the other cases, we consider a lift

$$\widetilde{B}_{\theta,\varphi,m}(x) = m\theta + x + \frac{1}{2\pi}\arg\left(\Gamma_m(x,\theta,\varphi)\right)$$

of $B_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ via $x \mapsto e^{2\pi i x}$. By Lemma 2.14,

$$\widetilde{B}_{\theta,\varphi,m}(x+1) = m\theta + x + 1 + \frac{1}{2\pi}\arg\left(\Gamma_m(x+1,\theta,\varphi)\right) = \widetilde{B}_{\theta,\varphi,m}(x) + 1$$

for all $x \in \mathbb{R}$. This implies that $B_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism. Consequently the rotation number of $\rho(\widetilde{B}_{\theta,\varphi,m})$ is well defined. By Lemma 2.15, we obtain that $\widetilde{B}^n_{1,\varphi,m}(x) = \widetilde{B}^n_{0,\varphi,m}(x) + mn$ and hence

$$\rho(\widetilde{B}_{1,\varphi,m}) = \rho(\widetilde{B}_{0,\varphi,m}) + m. \tag{13}$$

Moreover by Lemma 2.16, we obtain that $\widetilde{B}^n_{\theta,1,m}(x) = \widetilde{B}^n_{\theta,0,m}(x) + 2mn$ and hence

$$\rho(\widetilde{B}_{\theta,1,m}) = \rho(\widetilde{B}_{\theta,0,m}) + 2m. \tag{14}$$

These two equations (13) and (14) imply that

$$\rho(\widetilde{B}_{1,1,m}) = \rho(\widetilde{B}_{0,0,m}) + 3m.$$

Therefore in the case that $0 \le r < 1$, there exists $(\theta_0, \varphi_0) \in [0,1]^2$ such that

$$\alpha = \rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) \equiv \rho(\widetilde{B}_{\theta_0,\varphi_0,m}) \pmod 1$$

since the rotation number function $(\theta,\varphi) \mapsto \rho(B_{\theta,\varphi,m}|_{\mathbb{T}})$ is continuous. In the case that $r = 1$, if $2\varphi + \theta + \omega \equiv 0 \pmod 1$, then $\rho(B_{\theta,\varphi,m}|_{\mathbb{T}}) \equiv -m\omega \pmod 1$. Hence if $\alpha + m\omega \not\equiv 0 \pmod 1$, then there exists $(\theta_0, \varphi_0) \in [0,1]^2$ such that

$$\alpha = \rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) \equiv \rho(\widetilde{B}_{\theta_0,\varphi_0,m}) \pmod 1$$

and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod 1$. □

Remark 2.18. By theorem 2.9, the degree of $B_{\theta_0,\varphi_0,m}$ is $2m+1$.

Let m be a positive integer. We consider another Blaschke product

$$A(z) = e^{2\pi i\theta} z \left(\frac{z^m - a}{1 - \bar{a}z^m}\right)\left(\frac{z^m - b}{1 - \bar{b}z^m}\right)$$

of degree $2m+1$ with $a\bar{b} \neq 1$ and $0 \le |a| \le |b| < \infty$. Let $\lambda = abe^{2\pi i\theta}$ and let $\mu = \bar{a}\bar{b}e^{-2\pi i\theta}$. Then multipliers of fixed points $z = 0$ and $z = \infty$ are λ and μ respectively. Blaschke products

$$A(z) = e^{2\pi i\theta} z \left(\frac{z^m - a}{1 - \bar{a}z^m}\right)\left(\frac{z^m - b}{1 - \bar{b}z^m}\right)$$

and

$$B(z) = e^{2\pi i m\theta} z \left(\frac{z - a}{1 - \bar{a}z}\right)^m \left(\frac{z - b}{1 - \bar{b}z}\right)^m$$

are semiconjugate via $S_m(z) = z^m$, namely

$$B \circ S_m = S_m \circ A.$$

Therefore if $B|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism, then so is $A|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$. The following is a corollary of Theorem 2.17.

Corollary 2.19. *Let $\alpha \in [0,1]$ and let $\mu = re^{2\pi i\omega} \in \overline{\mathbb{D}}$. Besides let $a = a(\theta,\varphi)$ and $b = b(\theta,\varphi)$ be as in Theorem 2.9. Then for the Blaschke product*

$$A_{\theta,\varphi,m}(z) = e^{2\pi i\theta} z \left(\frac{z^m - a}{1 - \bar{a}z^m}\right)\left(\frac{z^m - b}{1 - \bar{b}z^m}\right),$$

$A_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism. Moreover

(a) *If $0 \le r < 1$, then there exists $(\theta_0,\varphi_0) \in [0,1]^2$ such that $\rho(A_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \alpha$.*
(b) *If $r = 1$ and $\alpha + \omega \not\equiv 0 \pmod 1$, then there exists $(\theta_0,\varphi_0) \in [0,1]^2$ such that $\rho(A_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \alpha$ and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod 1$.*

Proof. By Theorem 2.17, $B_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism. Therefore so is $A_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$. We consider a lift

$$\widetilde{A}_{\theta,\varphi,m}(x) = \theta + x + \frac{1}{2\pi}\arg\left\{\left(\frac{e^{2\pi imx} - a}{1 - \bar{a}\,e^{2\pi imx}}\right)\left(\frac{e^{2\pi imx} - b}{1 - \bar{b}\,e^{2\pi imx}}\right)\right\}$$

of $A_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ via $x \mapsto e^{2\pi ix}$. It is clear that $m\,\widetilde{A}_{\theta,\varphi,m}(x) = \widetilde{B}_{\theta,\varphi,m}(mx)$ and therefore

$$m\,\widetilde{A}^n_{\theta,\varphi,m}(x) = \widetilde{B}^n_{\theta,\varphi,m}(mx).$$

Since $A_{\theta,\varphi,m}|_{\mathbb{T}}$ and $B_{\theta,\varphi,m}|_{\mathbb{T}}$ are orientation-preserving homeomorphisms, we obtain that

$$m\,\rho(\widetilde{A}_{\theta,\varphi,m}) = \rho(\widetilde{B}_{\theta,\varphi,m})$$

and

$$m\,\rho(A_{\theta,\varphi,m}|_{\mathbb{T}}) = \rho(B_{\theta,\varphi,m}|_{\mathbb{T}}).$$

We consider the case that $0 \le r < 1$. By Theorem 2.17, for $\beta = m\alpha$ there exists $(\theta_0,\varphi_0) \in [0,1]^2$ such that

$$\rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \beta.$$

Therefore this implies that

$$\rho(A_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \alpha.$$

Next we consider the case that $r = 1$. It is clear that if $\alpha + \omega \not\equiv 0 \pmod 1$, then $\beta + m\omega \not\equiv 0 \pmod 1$. By Theorem 2.17, there exists $(\theta_0,\varphi_0) \in [0,1]^2$ such that

$$\rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \beta$$

and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod 1$. Hence we obtain that

$$\rho(A_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \alpha.$$

$\square$

2.3. **Quasiconformal surgery.**
In this subsection we prove Theorems 2.5 and 2.6 and Corollaries 2.7 and 2.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. If there exists $k \geq 1$ such that

$$\frac{1}{k} \leq \left| \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \right| \leq k$$

for all $x \in \mathbb{R}$ and all $t \geq 0$, then f is called *k-quasisymmetric*. A homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is *k-quasisymmetric* if its lift $\tilde{h} : \mathbb{R} \to \mathbb{R}$ is k-quasisymmetric.

Theorem 2.20 (Beurling-Ahlfors)**.** *Any k-quasisymmetric homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is extended to a K-quasiconformal map $F : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$. The dilatation K of F depends only on k.*

Hence if a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is k-quasisymmetric, then we can extend h to a K-quasiconformal map $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ whose dilatation depends only on k. The following theorem gives an equivalent condition for the rotation number and the linearizability of an orientation-preserving homeomorphism on the unit circle.

Theorem 2.21 (Herman-Świątek)**.** *The rotation number $\rho(f)$ of a real analytic orientation-preserving homeomorphism $f : \mathbb{T} \to \mathbb{T}$ is of bounded type if and only if f is quasisymmetrically linearizable, namely there exits a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ f \circ h^{-1}(z) = e^{2\pi i \rho(f)} z$.*

We recall that

$$E_{\lambda,\mu,m}(z) = z \left(\frac{z^m + \lambda}{\mu z^m + 1} \right), \quad F_{\lambda,\mu,m}(z) = z \left(\frac{z + \lambda}{\mu z + 1} \right)^m,$$

$$A_{\theta,\varphi,m}(z) = e^{2\pi i \theta} z \left(\frac{z^m - a}{1 - \bar{a} z^m} \right) \left(\frac{z^m - b}{1 - \bar{b} z^m} \right),$$

$$B_{\theta,\varphi,m}(z) = e^{2\pi i m \theta} z \left(\frac{z - a}{1 - \bar{a} z} \right)^m \left(\frac{z - b}{1 - \bar{b} z} \right)^m,$$

where $a = a(\theta, \varphi)$ and $b = b(\theta, \varphi)$ are the solutions of the equation (†). In this case, $z = e^{2\pi i \varphi}$ is a critical point of $B_{\theta,\varphi,m}$ and $z = e^{2\pi i (\varphi + j)/m}$ is a critical point of $A_{\theta,\varphi,m}$ for $j = 0, 1, \ldots, m-1$.

Proof of Theorem 2.5. By Corollary 2.19, there exist $(\theta, \varphi) \in [0,1]^2$ such that the degree of $A_{\theta,\varphi,m}$ is $2m+1$ and $\rho(A_{\theta,\varphi,m}|_{\mathbb{T}}) = \alpha$. By Theorem 2.21, there exists a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ A_{\theta,\varphi,m}|_{\mathbb{T}} \circ h^{-1} = R_\alpha$ since α is of bounded type, where $R_\alpha(z) = e^{2\pi i \alpha} z$. By the theorem of Beurling and Ahlfors, the quasisymmetric homeomorphism h has a quasiconformal extension $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ with $H(0) = 0$. We define a new map $\mathfrak{A}_{\theta,\varphi,m}$ as

$$\mathfrak{A}_{\theta,\varphi,m} = \begin{cases} A_{\theta,\varphi,m} & \text{on } \hat{\mathbb{C}} \setminus \mathbb{D}, \\ H^{-1} \circ R_\alpha \circ H & \text{on } \mathbb{D}. \end{cases}$$

The map $\mathfrak{A}_{\theta,\varphi,m}$ is quasiregular on $\hat{\mathbb{C}}$ since $\mathbb{T}$ is an analytic curve. Moreover $\mathfrak{A}_{\theta,\varphi,m}$ is a degree $m+1$ branched covering of $\hat{\mathbb{C}}$. We define a conformal structure $\sigma_{\theta,\varphi,m}$ as

$$\sigma_{\theta,\varphi,m} = \begin{cases} H^*(\sigma_0) & \text{on } \mathbb{D}, \\ \left(\mathfrak{A}^n_{\theta,\varphi,m}\right)^* \circ H^*(\sigma_0) & \text{on } \mathfrak{A}^{-n}_{\theta,\varphi,m}(\mathbb{D}) \setminus \mathbb{D} \text{ for all } n \in \mathbb{N}, \\ \sigma_0 & \text{on } \hat{\mathbb{C}} \setminus \bigcup_{n=1}^{\infty} \mathfrak{A}^{-n}_{\theta,\varphi,m}(\mathbb{D}), \end{cases}$$

where σ_0 is the standard conformal structure on $\hat{\mathbb{C}}$. The conformal structure $\sigma_{\theta,\varphi,m}$ is invariant under $\mathfrak{A}_{\theta,\varphi,m}$ and its maximal dilatation is the dilatation of H since H is quasiconformal and $A_{\theta,\varphi,m}$ is holomorphic. By the measurable Riemann mapping theorem, there exists a quasiconformal map $\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\Phi^*\sigma_0 = \sigma_{\theta,\varphi,m}$. Therefore $\Phi \circ \mathfrak{A}_{\theta,\varphi,m} \circ \Phi^{-1}$ is a rational function of degree $m+1$. We normalize Φ by $\Phi(0) = 0$, $\Phi(\infty) = \infty$ and $\Phi(b_1)$ is a point with $\Phi(b_1)^m = -\lambda$, where b_1 is an m-th root of b and $\lambda = e^{2\pi i\alpha}$. Then other m-th roots $b_2, \ldots, b_m$ of b also satisfy that $\Phi(b_j)^m = -\lambda$.

Lemma 2.22. *For the quasiconformal map Φ normalized as above,*

$$E_{\lambda,\mu,m} = \Phi \circ \mathfrak{A}_{\theta,\varphi,m} \circ \Phi^{-1}.$$

Proof of Lemma 2.22. First we consider the case that $\mu \neq 0$. Since orders of zeros and poles are invariant under conjugation, we obtain that

$$\Phi \circ \mathfrak{A}_{\theta,\varphi,m} \circ \Phi^{-1}(z) = \nu\, z \left(\frac{z^m + \lambda}{z^m + \xi}\right)$$

for some ν and ξ with $\nu\,\xi \neq 0$. Since multipliers of fixed points are also invariant under conjugation, we obtain that

$$\lambda = \left(\Phi \circ \mathfrak{A}_{\theta,\varphi,m} \circ \Phi^{-1}\right)'(0) = \nu \frac{\lambda}{\xi} \tag{15}$$

and

$$\mu = \frac{1}{\left(\Phi \circ \mathfrak{A}_{\theta,\varphi,m} \circ \Phi^{-1}\right)'(\infty)} = \frac{1}{\nu}. \tag{16}$$

By the equations (15) and (16), we obtain that $\nu = \xi = 1/\mu$. Therefore

$$\Phi \circ \mathfrak{A}_{\theta,\varphi,m} \circ \Phi^{-1}(z) = \frac{z}{\mu}\left(\frac{z^m + \lambda}{z^m + 1/\mu}\right) = z\left(\frac{z^m + \lambda}{\mu z^m + 1}\right) = E_{\lambda,\mu,m}(z).$$

In the case that $\mu = 0$, we obtain that

$$\Phi \circ \mathfrak{A}_{\theta,\varphi,m} \circ \Phi^{-1}(z) = \nu\, z\,(z^m + \lambda)$$

for some $\nu \neq 0$. Since

$$\lambda = \left(\Phi \circ \mathfrak{A}_{\theta,\varphi,m} \circ \Phi^{-1}\right)'(0) = \nu\lambda,$$

$\nu = 1$ and $\Phi \circ \mathfrak{A}_{\theta,\varphi,m} \circ \Phi^{-1} = E_{\lambda,\mu,m}$. ∎

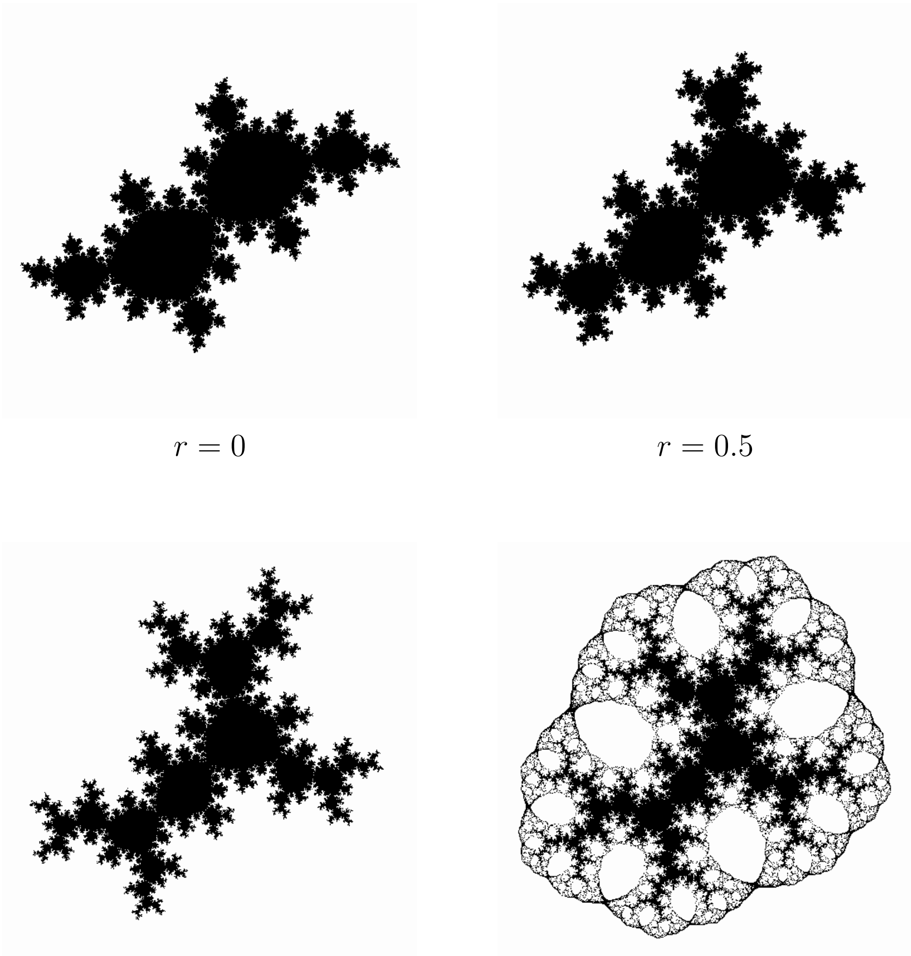

FIGURE 2. Golden Siegel disks of the rational function $F_{\lambda,\mu,1}$ centered at the origin, where $\lambda = e^{2\pi i\cdot(\sqrt{5}-1)/2}$ and $\mu = re^{2\pi i\cdot(\sqrt{5}-1)/2}$. In the case that $r = 1$, the point at infinity is the center of another golden Siegel disk.

The rational function $E_{\lambda,\mu,m}$ has a Siegel disk $\Delta = \Phi(\mathbb{D})$ with a critical point $\Phi(e^{2\pi i(\varphi+j)/m}) \in \partial\Delta$. Moreover $\partial\Delta = \Phi(\mathbb{T})$ is a quasicircle since Φ is quasiconformal. We have completed the proof of Theorem 2.5. $\square$

Remark 2.23. The boundary of the Siegel disk of $E_{\lambda,\mu,m}$ centered at the origin contains m critical points

$$\Phi(e^{2\pi i\varphi/m}),\ \Phi(e^{2\pi i(\varphi+1)/m}), \ldots,\ \Phi(e^{2\pi i(\varphi+m-1)/m}).$$

Proof of Theorem 2.6. Let λ be a complex number satisfying $\lambda^m = e^{2\pi i\alpha}$. By the assumption $e^{2\pi i\alpha}\mu^m \neq 1$, we obtain that $\lambda\mu \neq 1$. By Theorem 2.5, the boundary of the Siegel disk Δ of $E_{\lambda,\mu,m}$ centered at the origin is a quasicircle containing its critical point $\Phi(e^{2\pi i(\varphi+j)/m})$. Since two rational functions $E_{\lambda,\mu,m}$ and $F_{\lambda,\mu,m}$

are semiconjugate via $S_m(z) = z^m$, $S_m(\Delta)$ is the Siegel disk of $F_{\lambda,\mu,m}$ centered at the origin and the boundary $\partial S_m(\Delta)$ is a quasicircle containing its critical point $S_m(\Phi(e^{2\pi i(\varphi+j)/m}))$. □

Remark 2.24. The boundary of the Siegel disk of $F_{\lambda,\mu,m}$ centered at the origin contains only one critical point, since

$$S_m(\Phi(e^{2\pi i(\varphi+j)/m})) = S_m(\Phi(e^{2\pi i(\varphi+k)/m}))$$

for any j and k in $\{0, 1, \ldots, m-1\}$.

Proof of Corollary 2.7. Let $I(z) = 1/z$. Then $E_{\lambda,\mu,m} = I \circ E_{\mu,\lambda,m} \circ I$. Let Δ and Δ_∞ be Siegel disks of $E_{\lambda,\mu,m}$ centered at the origin and the point at infinity respectively. By Theorem 2.5, the boundary of Δ contains a critical point of $E_{\lambda,\mu,m}$. On the other hand, $I(\Delta_\infty)$ is the Siegel disk of $E_{\mu,\lambda,m}$ centered at the origin. By Theorem 2.5, the boundary of $I(\Delta_\infty)$ contains a critical point of $E_{\mu,\lambda,m}$. Therefore the boundary of Δ_∞ contains a critical point of $E_{\lambda,\mu,m}$. □

The proof of Corollary 2.8 is similar to that of Corollary 2.7.

3. Julia Sets of Quartic Polynomials and Polynomial Semigroups

For a polynomial of degree greater than one, the Julia set and the filled-in Julia set are either connected or else have uncountably many components. In the case that the Julia set of a quartic polynomial is neither connected nor totally disconnected, there exists a homeomorphism between the set of all components of the filled-in Julia set with the Hausdorff metric and some subset of the corresponding symbol space with the ordinary metric. Furthermore the quartic polynomial is topologically conjugate to the shift map via the homeomorphism. Moreover there exists a homeomorphism between the Julia set of the quartic polynomial and that of a certain polynomial semigroup.

3.1. **Homeomorphy.**

Definition 3.1. The *symbol space* on q symbols is the countable product $\Sigma_q = \{1, 2, \ldots, q\}^\omega$. For $s = (s_n)$ and $t = (t_n)$ in Σ_q, the metric ρ on Σ_q is defined as

$$\rho(s,t) = \sum_{n=0}^{\infty} \frac{\delta(s_n, t_n)}{2^n}, \quad \text{where} \quad \delta(k,l) = \begin{cases} 1 & \text{if } k \neq l, \\ 0 & \text{if } k = l. \end{cases}$$

Then (Σ_q, ρ) is a compact metric space. The *shift map* $\sigma : \Sigma_q \to \Sigma_q$ is defined as

$$\sigma((s_0, s_1, s_2, \ldots)) = (s_1, s_2, \ldots).$$

The shift map σ is continuous with respect to the metric ρ.

The connectivity of the Julia set of a polynomial is affected by the behavior of finite critical points. Dynamics on the Julia set is simple if its all finite critical points belong to the attracting basin of the point at infinity.

Theorem 3.2. *Let f be a polynomial of degree $d \geq 2$.*

- *If all finite critical points of f belong to the attracting basin $A(\infty)$, then the Julia set $J(f)$ is totally disconnected and correspond with the filled-in Julia set $K(f)$. Furthermore $f : J(f) \to J(f)$ is topologically conjugate to the shift map $\sigma : \Sigma_d \to \Sigma_d$.*
- *Both $J(f)$ and $K(f)$ are connected if and only if all finite critical points of f belong to $K(f)$.*

Definition 3.3. The triple (f, U, V) is a *polynomial-like map* of degree d if U and V are bounded simply connected domains such that $\overline{U} \subset V$ and $f : U \to V$ is a holomorphic proper map of degree d. The *filled-in Julia set* $K(f)$ of a polynomial-like map (f, U, V) is defined as

$$K(f) = \{z \in U : \{f^n(z)\}_{n=1}^{\infty} \subset U\}.$$

Definition 3.4. For a compact subset A in $\mathbb{C}$ and a positive number δ, let $A[\delta]$ be the δ-neighborhood of A. For compact subsets A and B in $\mathbb{C}$, we define the *Hausdorff metric* d_H as

$$d_H(A, B) = \inf \{\delta : A \subset B[\delta] \text{ and } B \subset A[\delta]\}.$$

Situation. Let f be a quartic polynomial and let c_1, c_2 and c_3 be finite critical points of f. Besides let G be the Green's function associated with the filled-in Julia set $K(f)$. We assume that $G(c_1) = G(c_2) = 0$ and $G(c_3) > 0$, namely c_1 and c_2 belong to $K(f)$ and c_3 belongs to $A(\infty)$.

Let U be the bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_3)))$. We assume that U_A and U_B are the different bounded components of $\mathbb{C} \setminus G^{-1}(G(c_3))$ such that $c_1 \in U_A$ and $c_2 \in U_B$. Then U_A and U_B are proper subsets of U. Furthermore $(f|_{U_A}, U_A, U)$ and $(f|_{U_B}, U_B, U)$ are polynomial-like maps of degree two. We set $f_1 = f|_{U_A}$ and $f_2 = f|_{U_B}$.

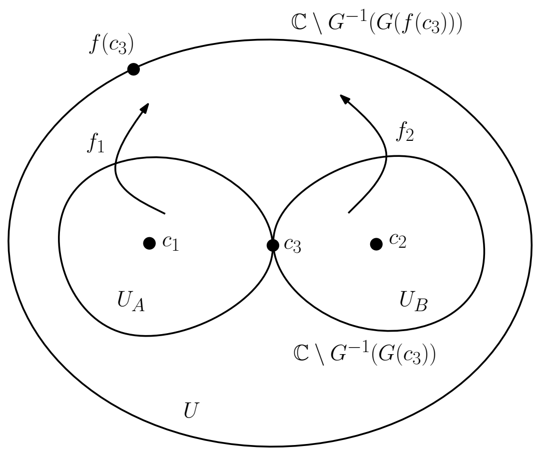

FIGURE 3. Polynomial-like maps (f_1, U_A, U) and (f_2, U_B, U).

Under this situation, we define the *A-B kneading sequence* $(\alpha_n)_{n \geq 0}$ of c_i as

$$\alpha_n = \begin{cases} A & \text{if } f^n(c_i) \in U_A, \\ B & \text{if } f^n(c_i) \in U_B. \end{cases}$$

We assume that the A-B kneading sequence of c_1 is $(AAA\cdots)$ and the A-B kneading sequence of c_2 is $(BBB\cdots)$. This implies that $K(f_1)$ and $K(f_2)$ are connected.

Let $K(f)^*$ be the set of all components of $K(f)$. Since $G(c_3) > 0$, the Julia set $J(f)$ and the filled-in Julia set $K(f)$ are disconnected and have uncountably many components respectively. Therefore $K(f)^*$ is an uncountable set and becomes a metric space with the Hausdorff metric d_H. We define a map $F : K(f)^* \to K(f)^*$ as $F(K) = f(K)$ for $K \in K(f)^*$. This map F is continuous with respect to the Hausdorff metric d_H.

Let $\Sigma_6 = \{1, 2, 3, 4, \mathsf{A}, \mathsf{B}\}^\omega$ be the symbol space which we treat mainly in this section. We define a subset Σ of Σ_6 as follows: A point $s = (s_n)$ belongs to Σ if and only if

(1) If $s_n = \mathsf{A}$, then $s_{n+1} = \mathsf{A}$.
(2) If $s_n = \mathsf{B}$, then $s_{n+1} = \mathsf{B}$.
(3) If $s_n = \mathsf{A}$ and $s_{n-1} \neq \mathsf{A}$, then $s_{n-1} = 3$ or 4.
(4) If $s_n = \mathsf{B}$ and $s_{n-1} \neq \mathsf{B}$, then $s_{n-1} = 1$ or 2.
(5) If $s \in \Sigma_4 = \{1, 2, 3, 4\}^\omega$, then there exist subsequences $(s_{n(k)})_{k=1}^\infty$ and $(s'_{n(l)})_{l=1}^\infty$ such that $s_{n(k)} = 1$ or 2 for all $k \geq 1$ and $s'_{n(l)} = 3$ or 4 for all $l \geq 1$.

It is the first goal of this section to prove the following theorem.

Theorem 3.5. *Let f be a quartic polynomial. We assume that its finite critical points $c_1, c_2 \in K(f)$ and $c_3 \in A(\infty)$ are all different and assume that $J(f)$ is disconnected but not totally disconnected. Moreover we assume that the A-B kneading sequence of c_1 is $(AAA\cdots)$ and the A-B kneading sequence of c_2 is $(BBB\cdots)$. Then there exists a homeomorphism $\Lambda : K(f)^* \to \Sigma$ such that $\Lambda \circ F = \sigma \circ \Lambda$.*

Remark 3.6. The property (5) of Σ is essential. For instance, a sequence

$$\left\{ s^{(n)} = (\underbrace{1, 1, \dots, 1}_{n\text{-times}}, \mathsf{B}, \mathsf{B}, \mathsf{B}, \dots) \right\}_{n=1}^{\infty}$$

in Σ converges to $s = (1, 1, 1, \dots)$ but s is not in Σ. Each $s^{(n)}$ corresponds to a component of backward iterated images of the filled-in Julia set $K(f_2)$. These backward components converge to a repelling fixed point in $\partial K(f_1)$. Therefore we can consider that the point $s = (1, 1, 1, \dots)$ corresponds to the repelling fixed point. However the fixed point is not a component of $K(f)$. Similarly a sequence like $(1, 2, 1, 2, \dots)$ corresponds to a periodic point of period two in $\partial K(f_1)$.

In the case of other critical configurations, we obtain similar results to Theorem 3.5 (see the subsection 3.3).

Theorem 3.7. *Let f be a quartic polynomial. We assume that its finite critical points c_1, c_2 and c_3 satisfy that $G(c_1) = 0$, $G(c_3) \geq G(c_2) > 0$ and $f^n(c_2) \neq c_3$ for all $n \geq 0$ and assume that $J(f)$ is disconnected but not totally disconnected. Moreover we assume that the kneading sequence of c_1 is $(CCC\cdots)$. Then there exist a subset Σ of $\Sigma_5 = \{1, 2, 3, 4, \mathsf{C}\}^\omega$ and a homeomorphism $\Lambda : K(f)^* \to \Sigma$ such that $\Lambda \circ F = \sigma \circ \Lambda$.*

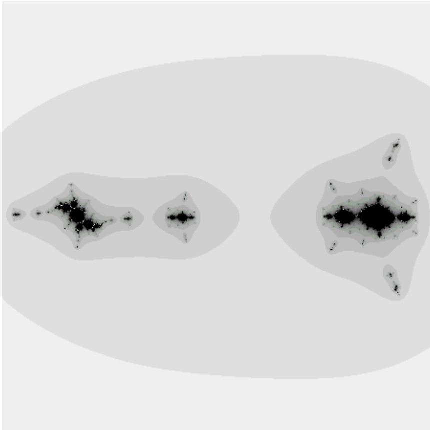

Figure 4. The filled-in Julia set of $f(z) = z^4 + cz^3 - (3c/2+2)z^2 + 2$, where $c = 0.594618 - 0.017361i$. Its finite critical points are $-1.445963 + 0.013021i$ and 1 in $K(f)$ and 0 in $A(\infty)$.

Theorem 3.8. *Let f be a quartic polynomial. We assume that its finite critical points c_1, c_2 and c_3 satisfy that $c_1 = c_2$, $c_1 \in K(f)$ and $c_3 \in A(\infty)$ and assume that $J(f)$ is disconnected but not totally disconnected. Moreover we assume that the kneading sequence of c_1 is $(BBB\cdots)$. Then there exist a subset Σ of $\Sigma_5 = \{1,2,3,4,\mathsf{B}\}^\omega$ and a homeomorphism $\Lambda : K(f)^* \to \Sigma$ such that $\Lambda \circ F = \sigma \circ \Lambda$.*

Figure 4 is an example of the situation described in Theorem 3.5. The biggest component of the left-hand side in Figure 4 is like the filled-in Julia set of some polynomial of degree two, and the biggest component of the right-hand side is also like the filled-in Julia set of some polynomial of degree two. In fact, these are the filled-in Julia sets of polynomial-like maps of degree two. Therefore we can consider that the quartic polynomial in this example is constructed from two polynomials of degree two. In general, we can consider that a quartic polynomial which satisfies the assumption of Theorem 3.5 is constricted from two polynomials of degree two. Under this consideration, we conjecture that the Julia set of the quartic polynomial is homeomorphic to that of some polynomial semigroup generated by two polynomials of degree two. This conjecture is actually correct.

Theorem 3.9. *Under the assumption of Theorem 3.5, there exist polynomials g_1 and g_2 of degree two and a homeomorphism h on $K(f)$ such that*

$$h(J(f)) = J(G),$$

where $G = \langle g_1, g_2 \rangle$ is a polynomial semigroup.

Similarly we obtain a similar result to Theorem 3.9.

Theorem 3.10. *Under the assumption of Theorem 3.7 or Theorem 3.8, there exist a polynomial semigroup G and a homeomorphism h on $K(f)$ such that*

$$h(J(f)) = J(G).$$

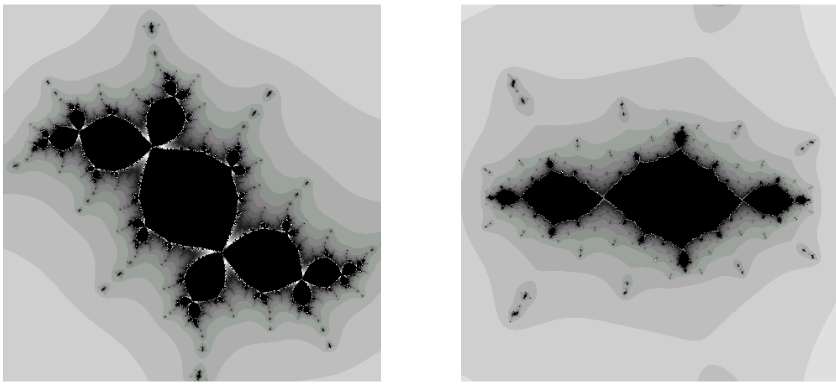

FIGURE 5. Enlargements of some parts of Figure 4. The biggest component of the left figure contains $-1.445963 + 0.013021i$ and the biggest component of the right figure contains 1.

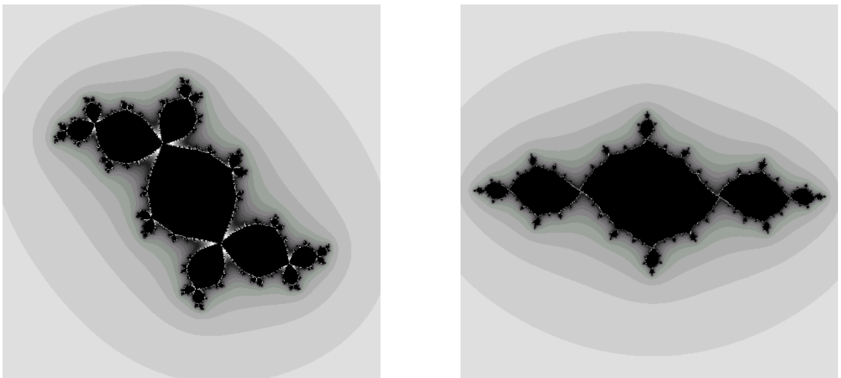

FIGURE 6. The filled-in Julia sets of $f_c(z) = z^2 + c$, where $c = -0.124444 + 0.711111i$ (left) and $c = -1.051111 + 0.060000i$ (right).

3.2. **Filled-in Julia sets and symbol spaces.**
Let W be the unbounded component of $\mathbb{C} \setminus G^{-1}(G(c_3))$. Its boundary ∂W contains c_3. Then a conformal map Ψ with the following properties exists: There exists $r > 1$ such that $\Psi : \mathbb{C} \setminus \overline{\mathbb{D}}_r \to W$ is a conformal isomorphism, where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. For a positive number t with $0 \leq t < 1$, $R(t) = \Psi(\{z \in \mathbb{C} : \arg(z) = 2\pi t$ and $|z| > r\})$ is called the *external ray* with angle t for $K(f)$.

Let R be the intersection of the external ray that passes through $f(c_3)$ and $\mathbb{C} \setminus \overline{U}$. Two of four rays $f^{-1}(R)$ have a limit point c_3 (see Figure 7). The set $\Psi^{-1}(f^{-1}(R))$ consists of four half-lines extended from $\partial \mathbb{D}_r$ with adjacent angles $\pi/2$ (see Figure 8). There are three invariant half-lines extended from the unit circle under the map $z \mapsto z^4$ and their angles are $0, 1/3$ and $2/3$. At least two of three invariant half-lines do not overlap with $\Psi^{-1}(f^{-1}(R))$. Let $\widetilde{R}_1$ be the intersection of one of these invariant half-lines and $\mathbb{C} \setminus \overline{\mathbb{D}}_r$. Let R_1 be the image of $\widetilde{R}_1$ under Ψ. We extend R_1 to become the invariant ray under f. Let R_0 be a component of $f^{-1}(R_1)$ which satisfies that $R_1 \cap R_0 \neq \emptyset$. Then $R_1 \subset R_0$ and f maps $J_0 = R_0 \setminus R_1$ onto

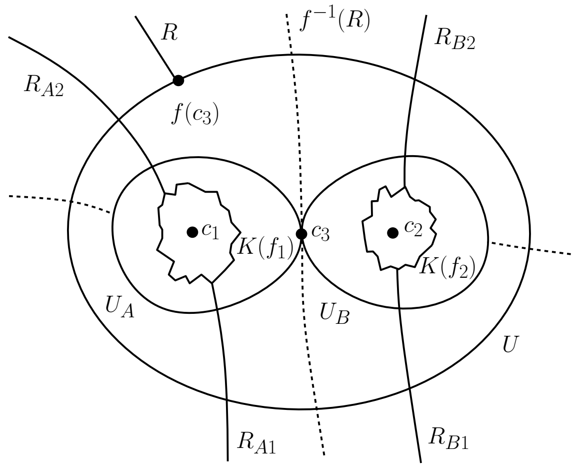

FIGURE 7. Some external rays. Dashed lines are $f^{-1}(R)$.

$J_1 = R_1 \cap \overline{U}$. Inductively, let R_{-n} be a component of $f^{-1}(R_{-(n-1)})$ which satisfies that $R_{-(n-1)} \cap R_{-n} \neq \emptyset$. Then $R_{-(n-1)} \subset R_{-n}$ and f maps J_{-n} onto $J_{-(n-1)}$, where

$$J_{-n} = \begin{cases} R_{-n} \setminus R_{-(n-1)} & \text{if } n \geq 0, \\ R_1 \cap \overline{U} & \text{if } n = -1. \end{cases}$$

The limit of this construction is the f-invariant ray

$$R_\infty = \bigcup_{n=0}^{\infty} R_{-n} = R_1 \cup \left(\bigcup_{n=0}^{\infty} J_{-n} \right).$$

Lemma 3.11 ([43, Lemma 5.2]). *Let F be a rational map and let X denote the closure of the union of the postcritical set and possible rotation domains of F. If $\gamma : (-\infty, 0] \to \hat{\mathbb{C}} \setminus X$ is a curve with*

$$F^{nk}(\gamma(-\infty, -k]) = \gamma(-\infty, 0]$$

for all positive integers k, then the limit

$$\lim_{t \to -\infty} \gamma(t)$$

exists and is a repelling or parabolic periodic point of F whose period divides n.

We can apply Lemma 3.11 to $R_\infty \setminus R_1 = \bigcup_{n=0}^{\infty} J_{-n}$, setting γ such that

$$\gamma(-(k+1), -k] = J_{-k}$$

for all positive integers k. Therefore R_∞ lands at a repelling or parabolic fixed point of f. If R_∞ lands at a point on $K(f_1)$, then we describe R_∞ with R_{A1}. Similarly if R_∞ lands at a point on $K(f_2)$, then we describe R_∞ with R_{B1}. If the angle of R_1 and $f^{-1}(R)$ is taken into consideration, we can obtain both R_{A1} and R_{B1} by choosing $\widetilde{R}_1$ well. Let R_{A2} and R_{B2} be components of $f^{-1}(R_{A1})$ and $f^{-1}(R_{B1})$ which satisfy that $R_{A2} \cap U_A \neq \emptyset$ and $R_{B2} \cap U_B \neq \emptyset$ and differ from R_{A1} and R_{B1}

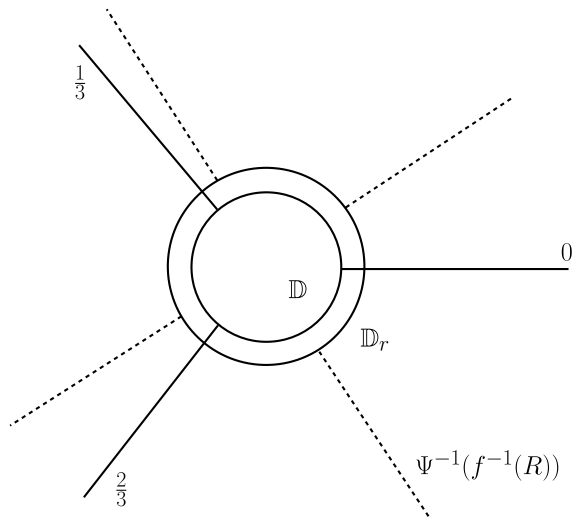

FIGURE 8. Solid lines are invariant under $z \mapsto z^4$ and dashed lines are $\Psi^{-1}(f^{-1}(R))$.

respectively. We set $V_A = U \setminus (K(f_1) \cup R_{A1})$ and $V_B = U \setminus (K(f_2) \cup R_{B1})$. Let I_1, I_2, I_3 and I_4 be inverse branches of f^{-1} such that

$$I_1 : V_A \to U_1, \quad I_2 : V_A \to U_2,$$
$$I_3 : V_B \to U_3, \quad I_4 : V_B \to U_4,$$

where U_1 and U_2 are components of $U_A \setminus (K(f_1) \cup R_{A1} \cup R_{A2})$. Similarly U_3 and U_4 are components of $U_A \setminus (K(f_2) \cup R_{B1} \cup R_{B2})$.

We define a map $\Lambda : K(f)^* \to \Sigma$ as follows: For K in $K(f)^*$,

$$[\Lambda(K)]_n = \begin{cases} i & \text{if} \quad f^n(K) \subset U_i, \\ \mathsf{A} & \text{if} \quad f^n(K) = K(f_1), \\ \mathsf{B} & \text{if} \quad f^n(K) = K(f_2), \end{cases}$$

where $i = 1, 2, 3, 4$ and $n \geq 0$.

Lemma 3.12. *The map $\Lambda : K(f)^* \to \Sigma$ is continuous.*

Proof. For any positive number ε, there exists a positive integer N such that $1/2^N < \varepsilon$. We take $K \in K(f)^*$ arbitrarily and set $s = \Lambda(K) = (s_0, s_1, \ldots, s_N, \ldots)$. We consider the case that $s \in \Sigma \cap \Sigma_4$ first. By the continuity of f, there exist positive numbers $\delta_1, \ldots, \delta_N$ such that $f^k(K[\delta_k]) \subset U_{s_k}$ for $k = 1, 2, \ldots, N$. Let δ be the minimum value of δ_k. Then $f^k(K[\delta]) \subset U_{s_k}$ for $k = 1, 2, \ldots, N$. Any component K' of $K(f)$ with $d_H(K, K') < \delta$ satisfies that $K' \subset K[\delta]$ by the definition of the Hausdorff metric. Moreover any component $K' \subset K[\delta]$ of $K(f)$ satisfies that $\Lambda(K') = (s_0, s_1, \ldots, s_N, t_{N+1}, \ldots)$. Therefore if any component K' of $K(f)$ satisfies that $d_H(K, K') < \delta$, then

$$\rho(\Lambda(K), \Lambda(K')) = \sum_{k=N+1}^{\infty} \frac{\delta(s_k, t_k)}{2^k} \leq \sum_{k=N+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^N} < \varepsilon.$$

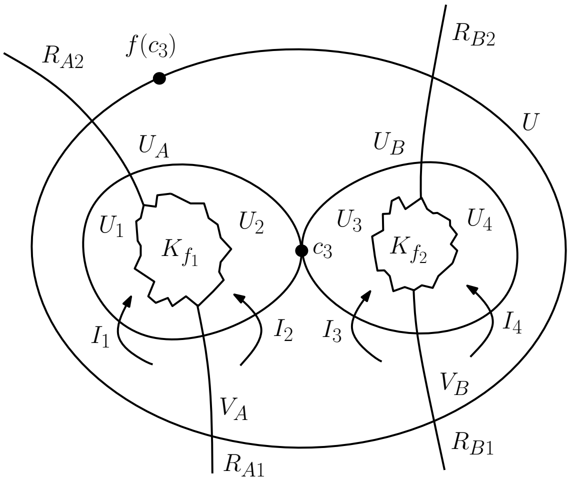

FIGURE 9. Inverse branches of f^{-1}.

If $s_n = \mathsf{A}$ and $s_{n-1} \neq \mathsf{A}$ or $s_n = \mathsf{B}$ and $s_{n-1} \neq \mathsf{B}$, then s is an isolated point in Σ. Since the corresponding K is also an isolated point in $K(f)^*$, the map Λ is continuous at K. □

We define a map $\widetilde{\Lambda} : \Sigma \to K(f)^*$ as follows: For $s = (s_n)$ in Σ, if $s_n = \mathsf{A}$ and $s_{n-1} \neq \mathsf{A}$, then we define $\widetilde{\Lambda}(s)$ as

$$\widetilde{\Lambda}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}} (K(f_1)) .$$

If $s_n = \mathsf{B}$ and $s_{n-1} \neq \mathsf{B}$, then we define $\widetilde{\Lambda}(s)$ as

$$\widetilde{\Lambda}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}} (K(f_2)) .$$

If s belongs to Σ_4, then there exists a subsequence $(s_{n(l)})_{l=1}^{\infty}$ such that $s_{n(l)} = 1$ or 2 and $s_{n(l)-1} = 3$ or 4. We set $K_s^{(l)} = I_{s_0} \circ \cdots \circ I_{s_{n(l)-1}}(\overline{U}_A)$ and then $K_s^{(l)} \supset K_s^{(l+1)}$. We define $\widetilde{\Lambda}(s)$ as

$$\widetilde{\Lambda}(s) = \bigcap_{l=1}^{\infty} K_s^{(l)}.$$

The set $\bigcap_{l=1}^{\infty} K_s^{(l)}$ is a one-point set since each I_k decreases the Poincaré distance on V_A or V_B.

Remark 3.13. We check that I_k decreases the Poincaré distance on V_A or V_B. For x and y in V_A, let γ be the Poincaré geodesic from x to y in V_A . Then there exists a constant $c < 1$ such that

$$\int_{I_1(\gamma)} ds_{V_A} \leq c \int_{I_1(\gamma)} ds_{U_1},$$

where ds_{V_A} and ds_{U_1} are the Poincaré metrics on V_A and U_1 respectively. Let γ' be the Poincaré geodesic from $I_1(x)$ to $I_1(y)$ in V_A. Then

$$\mathrm{dist}_{V_A}(I_1(x), I_1(y)) = \int_{\gamma'} ds_{V_A} \leq \int_{I_1(\gamma)} ds_{V_A},$$

where dist_{V_A} is the Poincaré distance on V_A. Since I_1 is conformal,

$$\int_{I_1(\gamma)} ds_{U_1} = \int_\gamma I_1^*(ds_{U_1}) = \int_\gamma ds_{V_A} = \mathrm{dist}_{V_A}(x,y).$$

Consequently we obtain that

$$\mathrm{dist}_{V_A}(I_1(x), I_1(y)) \leq c \cdot \mathrm{dist}_{V_A}(x,y).$$

Therefore I_1 decreases the Poincaré distance on V_A. Similarly we can show that I_2, I_3 and I_4 decrease the Poincaré distance on V_A or V_B.

Lemma 3.14. *The map $\widetilde{\Lambda}$ is the inverse map of Λ.*

Proof. We show that $\Lambda \circ \widetilde{\Lambda}$ and $\widetilde{\Lambda} \circ \Lambda$ are the identity maps. Let $s = (s_0, s_1, s_2, \ldots)$ be a point in Σ. If $s_n = \mathsf{A}$ and $s_{n-1} \neq \mathsf{A}$, then $\widetilde{\Lambda}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K(f_1))$. By definition of $\widetilde{\Lambda}$, we obtain that

$$f^k(\widetilde{\Lambda}(s)) = \begin{cases} I_{s_k} \circ \cdots \circ I_{s_{n-1}}(K(f_1)) \subset U_{s_k} & \text{if } 0 \leq k \leq n-1, \\ K(f_1) & \text{if } n \leq k. \end{cases}$$

Therefore $[\Lambda(\widetilde{\Lambda}(s))]_k = s_k$ and $\Lambda \circ \widetilde{\Lambda}(s) = s$. Similarly if $s_n = \mathsf{B}$ and $s_{n-1} \neq \mathsf{B}$, then $\Lambda \circ \widetilde{\Lambda}(s) = s$. If s belongs to Σ_4, then

$$f^k\Big(\widetilde{\Lambda}(s)\Big) = f^k\left(\bigcap_{l=1}^{\infty} K_s^{(l)}\right) \subset \bigcap_{l=1}^{\infty} f^k\Big(K_s^{(l)}\Big) \subset U_{s_k}.$$

Therefore $[\Lambda(\widetilde{\Lambda}(s))]_k = s_k$ and $\Lambda \circ \widetilde{\Lambda}(s) = s$. Consequently $\Lambda \circ \widetilde{\Lambda}$ is the identity map on Σ. It is clear that $\widetilde{\Lambda} \circ \Lambda$ is the identity map on $K(f)^*$. □

Lemma 3.15. *The map $\Lambda^{-1} : \Sigma \to K(f)^*$ is continuous.*

Proof. For any $s = (s_0, s_1, s_2, \ldots)$ in Σ, we set $K = \Lambda^{-1}(s)$. If $s_n = \mathsf{A}$ and $s_{n-1} \neq \mathsf{A}$, then $K = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K(f_1))$. Since K is an isolated point in $K(f)^*$, Λ^{-1} is continuous at s. Similarly if $s_n = \mathsf{B}$ and $s_{n-1} \neq \mathsf{B}$, then Λ^{-1} is continuous at s. We take a positive number ε arbitrarily. If s belongs to Σ_4, then

$$\Lambda^{-1}(s) = \bigcap_{l=1}^{\infty} K_s^{(l)}.$$

Since $K_s^{(l)} \supset K_s^{(l+1)}$ and $\Lambda^{-1}(s)$ is a one-point set, there exists $l_0 \geq 1$ such that

$$\Lambda^{-1}(s) \subset K_s^{(l_0)} \subset \Lambda^{-1}(s)[\varepsilon].$$

We set $\delta = 1/2^{n(l_0)-1}$ and consider a point t in Σ with $\rho(s,t) < \delta$. Then

$$t = (s_0, s_1, \ldots, s_{n_{l_0}-1}, s_{n_{l_0}}, t_{n_{l_0}+1}, \ldots).$$

By definition of $\Lambda^{-1}(t)$, if t belongs to $\Sigma \setminus \Sigma_4$, then

$$\Lambda^{-1}(t) \subset K_s^{(l_0)} \subset \Lambda^{-1}(s)[\varepsilon].$$

If t belongs to Σ_4, then
$$\Lambda^{-1}(t) = \bigcap_{l=1}^{\infty} K_t^{(l)}.$$
In this case, it is clear that $K_t^{(l)} = K_s^{(l)}$ for $l = 1, 2, \ldots, l_0$. Therefore we obtain that
$$\Lambda^{-1}(t) \subset K_s^{(l_0)} \subset \Lambda^{-1}(s)[\varepsilon].$$
Since $\Lambda^{-1}(s)$ is a one-point set, for a point t in Σ with $\rho(s,t) < \delta$,
$$d_H(\Lambda^{-1}(s), \Lambda^{-1}(t)) = \inf\{\varepsilon' : \Lambda^{-1}(t) \subset \Lambda^{-1}(s)[\varepsilon']\} < \varepsilon.$$
Therefore Λ^{-1} is continuous at s. □

Lemma 3.16. *Two maps F and σ are topologically conjugate via the homeomorphism Λ, namely $\Lambda \circ F = \sigma \circ \Lambda$.*

Proof. For a point K in $K(f)^*$, we set $\Lambda(K) = (s_0, s_1, s_2, \ldots)$. Then $\sigma \circ \Lambda(K) = (s_1, s_2, \ldots)$. On the other hand, $\Lambda \circ F(K) = \Lambda(f(K)) = (s_1, s_2, \ldots)$. Therefore $\Lambda \circ F = \sigma \circ \Lambda$. □

We have completed the proof of Theorem 3.5.

3.3. **Other critical configurations.**

For a quartic polynomial, the following two cases are also considered. We can similarly show Theorems 3.7 and 3.8. We assume that the Julia set is disconnected but not totally disconnected.

Case 1. Let f be a quartic polynomial and let c_1, c_2 and c_3 be finite critical points of f. We assume that $G(c_1) = 0$ and $G(c_3) \geq G(c_2) > 0$, namely c_1 belongs to $K(f)$ and c_2 and c_3 belong to $A(\infty)$. Moreover we assume that $f^n(c_2) \neq c_3$ for all $n \geq 0$.

Let U be the bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_2)))$. We assume that U_A, U_B and U_C are the different bounded components of $\mathbb{C} \setminus G^{-1}(G(c_2))$ such that $c_1 \in U_C$. Then U_A, U_B and U_C are proper subsets of U. Furthermore $(f|_{U_A}, U_A, U)$ and $(f|_{U_B}, U_B, U)$ are polynomial-like maps of degree one and $(f|_{U_C}, U_C, U)$ is a polynomial-like map of degree two.

Under this situation, we define the kneading sequence $(\alpha_n)_{n\geq 0}$ of c_1 as
$$\alpha_n = \begin{cases} A & \text{if } f^n(c_1) \in U_A, \\ B & \text{if } f^n(c_1) \in U_B, \\ C & \text{if } f^n(c_1) \in U_C. \end{cases}$$
We assume that the kneading sequence of c_1 is $(CCC\cdots)$.

Let $\Sigma_5 = \{1, 2, 3, 4, \mathsf{C}\}^\omega$ be the symbol space on five symbols. We define a subset Σ of Σ_5 as follows: A point $s = (s_n)$ belongs to Σ if and only if

(1) If $s_n = \mathsf{C}$, then $s_{n+1} = \mathsf{C}$.
(2) If $s_n = \mathsf{C}$ and $s_{n-1} \neq \mathsf{C}$, then $s_{n-1} = 1$ or 2.
(3) If $s \in \Sigma_4 = \{1, 2, 3, 4\}^\omega$, then there exists a subsequence $(s_{n(k)})_{k=1}^{\infty}$ such that $s_{n(k)} = 1$ or 2 for all $k \geq 1$.

Case 2. Let f be a quartic polynomial and let c_1, c_2 and c_3 be finite critical points of f such that $c_1 = c_2 \neq c_3$. We assume that $G(c_1) = 0$ and $G(c_3) > 0$, namely c_1 belongs to $K(f)$ and c_3 belongs to $A(\infty)$.

Let U be the bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_3)))$. We assume that U_A and U_B are the different bounded components of $\mathbb{C} \setminus G^{-1}(G(c_3))$ such that $c_1 \in U_B$. Then U_A and U_B are proper subsets of U. Furthermore $(f|_{U_A}, U_A, U)$ is a polynomial-like map of degree one and $(f|_{U_B}, U_B, U)$ is a polynomial-like map of degree three. We assume that the kneading sequence of c_1 is $(BBB\cdots)$.

Let $\Sigma_5 = \{1, 2, 3, 4, \mathsf{B}\}^\omega$ be the symbol space on five symbols. We define a subset Σ of Σ_5 as follows: A point $s = (s_n)$ belongs to Σ if and only if

(1) If $s_n = \mathsf{B}$, then $s_{n+1} = \mathsf{B}$.
(2) If $s_n = \mathsf{B}$ and $s_{n-1} \neq \mathsf{B}$, then $s_{n-1} = 1$.
(3) If $s \in \Sigma_4 = \{1, 2, 3, 4\}^\omega$, then there exists a subsequence $(s_{n(k)})_{k=1}^\infty$ such that $s_{n(k)} = 1$ for all $k \geq 1$.

3.4. **Julia sets of Polynomial Semigroups.**

Definition 3.17. A *rational semigroup* G is a semigroup generated by a family of non-constant rational functions $\{g_1, g_2, \dots, g_n, \dots\}$ defined on $\hat{\mathbb{C}}$. We denote this situation by

$$G = \langle g_1, g_2, \dots, g_n, \dots \rangle.$$

A rational semigroup G is called a *polynomial semigroup* if each $g \in G$ is a polynomial.

Definition 3.18. Let G be a rational semigroup. The *Fatou set* $F(G)$ of G is defined as

$$F(G) = \{z \in \hat{\mathbb{C}} : G \text{ is normal in a neighborhood of } z\}.$$

Its complement $\hat{\mathbb{C}} \setminus F(G)$ is called the *Julia set* $J(G)$ of G.

Henceforth, we prove Theorem 3.9. The following theorem on polynomial-like maps is important.

Theorem 3.19 ([15, 36])**.** *For every polynomial-like map (f, U, V) of degree $d \geq 2$ there exist a polynomial p of degree d, a neighborhood W of $K(f)$ in U and a quasiconformal map $h : W \to h(W)$ such that*

(a) *$h(K(f)) = K(p)$,*
(b) *the complex dilatation μ_h of h is zero almost everywhere on $K(f)$,*
(c) *$h \circ f = p \circ h$ on $W \cap f^{-1}(W)$.*

If $K(f)$ is connected, p is unique up to conjugation by affine map.

Under the assumptions of Theorem 3.5, the triples (f_1, U_A, U) and (f_2, U_B, U) are polynomial-like maps of degree two. Furthermore $K(f_1)$ and $K(f_2)$ are connected. By Theorem 3.19, there exist polynomials g_1 and g_2 of degree two with $K(g_1) \cap K(g_2) = \emptyset$, a neighborhood W_1 of $K(f_1)$ in U_A, a neighborhood W_2 of $K(f_2)$ in U_B and quasiconformal maps h_1 on W_1 and h_2 on W_2 such that $h_1(K(f_1)) = K(g_1)$ and $h_2(K(f_2)) = K(g_2)$.

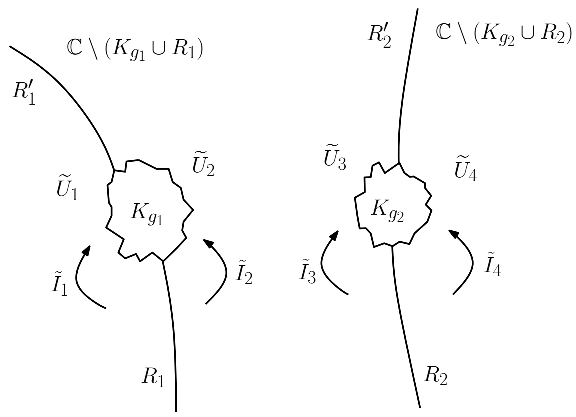

FIGURE 10. Inverse branches of g_1^{-1} and g_2^{-1}.

We define inverse branches $\tilde{I}_1$ and $\tilde{I}_2$ of g_1^{-1}. Since $K(g_1)$ is connected, there exists a conformal map $\Psi_1 : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K(g_1)$ such that $\Psi_1^{-1} \circ g_1 \circ \Psi_1(z) = z^2$. The external ray $R_1 = \Psi_1(\{z \in \mathbb{C} : \arg(z) = 0 \text{ and } |z| > 1\})$ lands at a fixed point of g_1. Let R_1' be the external ray which satisfies that $g_1(R_1') = R_1$ and differs from R_1. We replace g_2 so that

$$R_1 \cap K(g_2) = \emptyset \text{ and } R_1' \cap K(g_2) = \emptyset.$$

Then we define inverse branches $\tilde{I}_1$ and $\tilde{I}_2$ of g_1^{-1} as

$$\tilde{I}_1 : \mathbb{C} \setminus (K(g_1) \cup R_1) \to \widetilde{U}_1 \text{ and } \tilde{I}_2 : \mathbb{C} \setminus (K(g_1) \cup R_1) \to \widetilde{U}_2,$$

where $\widetilde{U}_1$ and $\widetilde{U}_2$ are components of $\mathbb{C} \setminus (K(g_1) \cup R_1 \cup R_1')$. Similarly, we can take external rays R_2 and R_2'. Then we define inverse branches $\tilde{I}_3$ and $\tilde{I}_4$ of g_2^{-1} as

$$\tilde{I}_3 : \mathbb{C} \setminus (K(g_2) \cup R_2) \to \widetilde{U}_3 \text{ and } \tilde{I}_4 : \mathbb{C} \setminus (K(g_2) \cup R_2) \to \widetilde{U}_4,$$

where $\widetilde{U}_3$ and $\widetilde{U}_4$ components of $\mathbb{C} \setminus (K(g_2) \cup R_2 \cup R_2')$.

For a point s in Σ, we set $K_s = \Lambda^{-1}(s)$ and $J_s = \partial K_s$. Then K_s is a component of $K(f)$ and J_s is a component of $J(f)$. For a point $s = (s_0, s_1, s_2, \ldots)$ in $\Sigma \setminus \Sigma_4$, we define a quasiconformal map h_s on a neighborhood of K_s. Let n be a non-negative number with $s_n = \mathsf{A}$ and $s_{n-1} \neq \mathsf{A}$ or $s_n = \mathsf{B}$ and $s_{n-1} \neq \mathsf{B}$. Then h_s is defined on $W_s = I_{s_o} \circ \cdots \circ I_{s_{n-1}}(W_i)$ as

$$h_s = \tilde{I}_{s_o} \circ \cdots \circ \tilde{I}_{s_{n-1}} \circ h_i \circ f^n, \text{ where } i = \begin{cases} 1 & \text{if } s_n = \mathsf{A} \text{ and } s_{n-1} \neq \mathsf{A}, \\ 2 & \text{if } s_n = \mathsf{B} \text{ and } s_{n-1} \neq \mathsf{B}. \end{cases}$$

We set $\widetilde{K}_s = h_s(K_s)$, $\widetilde{J}_s = \partial \widetilde{K}_s$ and $G = \langle g_1, g_2 \rangle$. If necessary, we replace g_1 and g_2 so that each $\widetilde{K}_s$ is disjoint. Since $\widetilde{J}_s = \partial \widetilde{K}_s = h_s(\partial K_s) = h_s(J_s)$ and $J(G)$ is backward invariant, h_s maps J_s onto a component $\widetilde{J}_s$ of $J(G)$. By definition, it is clear that $h_{(\mathsf{A},\mathsf{A},\mathsf{A},\ldots)} = h_1$ and $h_{(\mathsf{B},\mathsf{B},\mathsf{B},\ldots)} = h_2$.

Next, we define a homeomorphism

$$h : \bigcup_{s\in\Sigma\setminus\Sigma_4} K_s \to \bigcup_{s\in\Sigma\setminus\Sigma_4} \widetilde{K}_s$$

as $h|_{K_s} = h_s$.

Remark 3.20. For a point $s = (s_0, s_1, s_2, \ldots)$ in $\Sigma \cap \Sigma_4$, a one-point component K_s of $K(f)$ is characterized by using the Hausdorff topology. We set

$$t^{(n)} = \begin{cases} (s_0, s_1, \ldots, s_{n-1}, \mathsf{A}, \mathsf{A}, \mathsf{A}, \ldots) & \text{if } s_{n-1} = 3 \text{ or } 4, \\ (s_0, s_1, \ldots, s_{n-1}, \mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots) & \text{if } s_{n-1} = 1 \text{ or } 2. \end{cases}$$

Then the sequence $\{t^{(n)}\}_{n=1}^{\infty}$ belongs to $\Sigma \setminus \Sigma_4$ and $t^{(n)} \to s$ as $n \to \infty$. Since Λ^{-1} is continuous,

$$K_s = \Lambda^{-1}(s) = \lim_{n\to\infty} \Lambda^{-1}\left(t^{(n)}\right) = \lim_{n\to\infty} K_{t^{(n)}}.$$

Finally, we extend h homeomorphically on $K(f) = \bigcup_{s\in\Sigma} K_s$. For a point s in $\Sigma \cap \Sigma_4$, we define $\widetilde{K}_s = h(K_s)$ as

$$h(K_s) = \lim_{n\to\infty} h(K_{t^{(n)}}).$$

Then h is a homeomorphism between $K(f) = \bigcup_{s\in\Sigma} K_s$ and $\bigcup_{s\in\Sigma} \widetilde{K}_s$.

Lemma 3.21. *The Julia set $J(G)$ corresponds with the boundary of $\bigcup_{s\in\Sigma} \widetilde{K}_s$.*

Proof. Lemma 3.21 follows from the following.

Lemma 3.22 ([22])**.** *If z belongs to $J(G) \setminus E(G)$, then*

$$\overline{O^-(z)} = J(G),$$

where $O^-(z) = \{w \in \hat{\mathbb{C}} : \text{there exists } g \in G \text{ such that } g(w) = z\}$ *is the backward orbit of z and* $E(G) = \{z \in \hat{\mathbb{C}} : O^-(z) \text{ contains at most two points}\}$ *is the exceptional set of G.*

By Lemma 3.22,

$$\partial\left(\bigcup_{s\in\Sigma} \widetilde{K}_s\right) = \bigcup_{s\in\Sigma} \partial\widetilde{K}_s = \bigcup_{s\in\Sigma} \widetilde{J}_s = \overline{\bigcup_{s\in\Sigma\setminus\Sigma_4} \widetilde{J}_s} = J_G.$$

□

We have completed the proof of Theorem 3.9.

3.5. **Topologies of the symbol space.**
Theorem 3.5 means that componentwise dynamics of f on $K(f)$ can be simplified as dynamics of the shift map on Σ. The space (Σ, ρ) is not compact as the following example shows. The sequence

$$\left\{ s^{(n)} = (\underbrace{1, 1, \ldots, 1}_{n\text{-times}}, \mathsf{B}, \mathsf{B}, \mathsf{B}, \ldots) \right\}_{n=0}^{\infty}$$

in Σ converges to $s = (1, 1, 1, \ldots)$ but s is not in Σ. Since (Σ, ρ) is not compact, although the dynamical system $(K(f)^*, F)$ is conjugate to (Σ, σ) by Theorem 3.5, many good properties of the symbolic dynamical system are not available. So we impose a question: Is it possible to introduce a new topology on Σ which makes Σ compact and reflects the dynamical system $(K(f)^*, F)$ in a natural way? In this subsection, we answer this question.

Theorem 3.23. *There exists a topology $\mathcal{O}$ of Σ such that $(\Sigma, \mathcal{O})$ is compact, metrizable, perfect and totally disconnected. Moreover the shift map $\sigma : (\Sigma, \mathcal{O}) \to (\Sigma, \mathcal{O})$ is continuous.*

Regarding Λ in Theorem 3.5 just as a bijection between the sets $K(f)^*$ and Σ, we define $\mathcal{G}$ to be the quotient topology of $K(f)^*$ relative to Λ^{-1} and the topology $\mathcal{O}$ of Σ as in Theorem 3.23. Then $\Lambda : (K(f)^*, \mathcal{G}) \to (\Sigma, \mathcal{O})$ is a homeomorphism such that $\Lambda \circ F = \sigma \circ \Lambda$.

Corollary 3.24. *The topological space $(K(f)^*, \mathcal{G})$ is compact, metrizable, perfect and totally disconnected. Moreover $F : (K(f)^*, \mathcal{G}) \to (K(f)^*, \mathcal{G})$ is continuous.*

Known results in general topology.
We introduce some definitions and results in general topology. We refer to [29] and [37]. Let X be a topological space.

Definition 3.25. The topological space X is *sequentially compact* if every sequence of points of X contains a convergent subsequence. The topological space X is *countably compact* if every countable open covering of X has a finite subcovering. The topological space X is a *Lindelöf space* if every open covering of X has a countable subcovering.

Theorem 3.26. *If X is sequentially compact, then X is countably compact.*

Theorem 3.27. *If X satisfies the second axiom of countability, then X is a Lindelöf space.*

Theorem 3.28. *The topological space X is compact if and only if X is a countably compact Lindelöf space.*

Definition 3.29. The topological space X is a T_1*-space* if for any distinct points x and y in X, there exists an open neighborhood U of x such that $y \notin U$. The topological space X is a T_2*-space* or a *Hausdorff space* if for any distinct points x and y in X, there exist open neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

Definition 3.30. A T_1-space X is a *regular space* if for any x in X and any closed set L with $x \notin L$, there exists open neighborhoods U of x and V of L such that $U \cap V = \emptyset$.

Definition 3.31. A T_1-space X is a *normal space* if for any closed sets A and B of X with $A \cap B = \emptyset$, there exists open neighborhoods U of A and V of B such that $U \cap V = \emptyset$.

Theorem 3.32. *Each compact Hausdorff space is normal.*

Theorem 3.33. *The topological space X satisfying the second axiom of countability is metrizable if and only if X is a regular space.*

Definition 3.34. The topological space X is a 0-*dimensional space* if there exists an open basis $\mathcal{B}$ of X such that every $B \in \mathcal{B}$ is open and closed.

Theorem 3.35. *Let X be a compact Hausdorff space. The topological space X is a 0-dimensional space if and only if X is totally disconnected.*

Another topology of the symbol space.
We define a topology of Σ. Let k be a non-negative integer. If $s = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ belongs to Σ, then we define a subset $N_s^{(k)}$ of Σ as

$$N_s^{(k)} = \{s\} \cup \{t = (t_n) \in \Sigma : t_n = 1 \text{ or } 2 \text{ for } n \le k\}.$$

Similarly, if $s = (\mathsf{B}, \mathsf{B}, \mathsf{B}, \dots)$ belongs to Σ, then

$$N_s^{(k)} = \{s\} \cup \{t = (t_n) \in \Sigma : t_n = 3 \text{ or } 4 \text{ for } n \le k\}.$$

If $s = (s_0, \dots, s_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ belongs to Σ with $s_l \neq \mathsf{A}$, then

$$N_s^{(k)} = \{s\} \cup \left\{ t = (t_n) \in \Sigma : t_n = \begin{cases} s_n & \text{if } n \le l, \\ 1 \text{ or } 2 & \text{if } l+1 \le n \end{cases} \quad \text{for } n \le k \right\}.$$

Similarly, if $s = (s_0, \dots, s_l, \mathsf{B}, \mathsf{B}, \mathsf{B}, \dots)$ belongs to Σ with $s_l \neq \mathsf{B}$, then

$$N_s^{(k)} = \{s\} \cup \left\{ t = (t_n) \in \Sigma : t_n = \begin{cases} s_n & \text{if } n \le l, \\ 3 \text{ or } 4 & \text{if } l+1 \le n \end{cases} \quad \text{for } n \le k \right\}.$$

Finally, if $s = (s_n)$ belongs to $\Sigma \cap \Sigma_4$,

$$N_s^{(k)} = \{t = (t_n) \in \Sigma : t_n = s_n \text{ for } n \le k\}.$$

Then it holds that $N_s^{(k+1)} \subset N_s^{(k)}$ for all $s \in \Sigma$ and $k \ge 0$. Let $\mathcal{N}(s) = \{N_s^{(k)}\}_{k=0}^{\infty}$ and let $\mathcal{N} = \{\mathcal{N}(s) : s \in \Sigma\}$.

Lemma 3.36. *The set $\mathcal{N}$ is a neighborhood system of Σ.*

Proof. Let s be a point in Σ.

(i) If $N \in \mathcal{N}(s)$, then $s \in N$.
(ii) For N_1 and N_2 in $\mathcal{N}(s)$, there exist non-negative integers k_1 and k_2 such that $N_1 = N_s^{(k_1)}$ and $N_2 = N_s^{(k_2)}$. Let $N_3 = N_s^{(k)}$, where $k \ge \max\{k_1, k_2\}$. Then $N_3 \in \mathcal{N}(s)$ and $N_3 \subset N_1 \cap N_2$.

(iii) For N in $\mathcal{N}(s)$, there exists $k \geq 0$ such that $N = N_s^{(k)}$. For $t \in N$, we set $N' = N_t^{(k)}$. Then $N' \in \mathcal{N}(t)$ and $N' \subset N$.

Therefore $\mathcal{N}$ is a neighborhood system of Σ. □

Then $(\Sigma, \mathcal{O})$ is a topological space, where $\mathcal{O}$ is the topology generated by $\mathcal{N}$. We obtain immediately the following lemmas.

Lemma 3.37. *The topological space $(\Sigma, \mathcal{O})$ satisfies the first axiom of countability.*

Proof. We choose a neighborhood basis of $s \in \Sigma$ as $\mathcal{N}(s)$. □

Lemma 3.38. *The topological space $(\Sigma, \mathcal{O})$ is a Hausdorff space.*

Proof. For distinct points $s = (s_n)$ and $t = (t_n)$ in Σ, there exist $k \geq 0$ such that $s_k \neq t_k$. Let $M = N_s^{(k)} \in \mathcal{N}(s)$ and let $N = N_t^{(k)} \in \mathcal{N}(t)$. Then $M \cap N = \emptyset$. □

Lemma 3.39. *The topological space $(\Sigma, \mathcal{O})$ is perfect.*

Proof. For any s in Σ and any neighborhood $O \in \mathcal{O}$ of s, there exists $N \in \mathcal{N}(s)$ such that $N \subset O$. It is clear that $(O \setminus \{s\}) \cap \Sigma \supset (N \setminus \{s\}) \cap \Sigma \neq \emptyset$. □

We show that $(\Sigma, \mathcal{O})$ is compact. By Theorems 3.26, 3.27 and 3.28, we need only to show that $(\Sigma, \mathcal{O})$ is sequentially compact and satisfies the second axiom of countability.

Lemma 3.40. *The topological space $(\Sigma, \mathcal{O})$ is sequentially compact.*

Proof. Let $\{s^{(k)} = (s_n^{(k)})\}_{k=1}^{\infty} \subset \Sigma$. We choose a subsequence $\{s^{\langle\alpha\rangle}\}_{\alpha=0}^{\infty}$ as follows:
Step 0. There exists a subsequence $\{s^{(k_l)}\}_{l=1}^{\infty}$ such that $s_0^{(k_l)} = s_0$ for $l \geq 1$, where $s_0 = 1, 2, 3$ or 4. Let $s^{\langle 0\rangle}$ be one of $s^{(k_l)}$. Then $s^{\langle 0\rangle} = (s_0, s_1^{\langle 0\rangle}, s_2^{\langle 0\rangle}, \dots)$. We rewrite $s^{(k_l)}$ as $s^{(k)}$.
Step 1. There exists a subsequence $\{s^{(k_l)}\}_{l=1}^{\infty}$ such that $s_1^{(k_l)} = s_1$ for $l \geq 1$, where $s_1 = 1, 2, 3$ or 4. Let $s^{\langle 1\rangle}$ be one of $s^{(k_l)}$. Then $s^{\langle 1\rangle} = (s_0, s_1, s_2^{\langle 1\rangle}, s_3^{\langle 1\rangle}, \dots)$. We rewrite $s^{(k_l)}$ as $s^{(k)}$.
Step α**.** Inductively, we can choose $s^{\langle\alpha\rangle} = (s_0, \dots, s_\alpha, s_{\alpha+1}^{\langle\alpha\rangle}, s_{\alpha+2}^{\langle\alpha\rangle}, \dots)$.
Let $s = (s_0, s_1, s_2 \dots)$. If $s \in \Sigma$, then for a neighborhood $O \in \mathcal{O}$ of s, there exists $N = N_s^{(\alpha_0)} \in \mathcal{N}(s)$ such that $N \subset O$. If $\alpha \geq \alpha_0$, then $s^{\langle\alpha\rangle} \in N \subset O$. Therefore $s^{\langle\alpha\rangle}$ converges to s with respect to $\mathcal{O}$. If $s \notin \Sigma$, then there exists a unique $\beta \geq 0$ such that

(i) $s_{\beta-1} = 3$ or 4 and $s_n = 1$ or 2 for $\beta \leq n$,
(ii) $s_{\beta-1} = 1$ or 2 and $s_n = 3$ or 4 for $\beta \leq n$.

If (i) is the case, let

$$t = \begin{cases} (\mathsf{A}, \mathsf{A}, \mathsf{A}, \dots) & \text{if } \beta = 0, \\ (s_0, \dots, s_{\beta-1}, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots) & \text{if } \beta \geq 1. \end{cases}$$

If (ii) is the case, let

$$t = \begin{cases} (\mathsf{B}, \mathsf{B}, \mathsf{B}, \dots) & \text{if } \beta = 0, \\ (s_0, \dots, s_{\beta-1}, \mathsf{B}, \mathsf{B}, \mathsf{B}, \dots) & \text{if } \beta \geq 1. \end{cases}$$

Then we can prove that $s^{\langle\alpha\rangle}$ converges to t with respect to $\mathcal{O}$ by the same argument. Therefore $(\Sigma, \mathcal{O})$ is sequentially compact. □

Lemma 3.41. *The topological space $(\Sigma, \mathcal{O})$ satisfies the second axiom of countability.*

Proof. Let

$$\mathcal{B} = \bigcup_{s \in \Sigma \setminus \Sigma_4} \mathcal{N}(s).$$

First we show that $\mathcal{B}$ is an open basis of $(\Sigma, \mathcal{O})$. It is clear that $\mathcal{B} \subset \mathcal{O}$. Let $O \in \mathcal{O}$ and $s = (s_n) \in O$. If $s \in \Sigma \setminus \Sigma_4$, then there exists $M \in \mathcal{N}(s) \subset \mathcal{B}$ such that $s \in M \subset O$ by the definition of $\mathcal{O}$. If $s \in \Sigma \cap \Sigma_4$, then there exists $N = N_s^{(k)} \in \mathcal{N}(s)$ such that $s \in N \subset O$ by the definition of $\mathcal{O}$. However we do not know yet whether $N \in \mathcal{B}$ at this stage. Let $t = (s_0, \dots, s_k, t_{k+1}, t_{k+2}, \dots) \in N \cap (\Sigma \setminus \Sigma_4)$ and let $M = N_t^{(k)}$. Then $M \in \mathcal{B}$ and, in fact, $M = N$. Therefore $s \in M = N \subset O$. Consequently $\mathcal{B}$ is an open basis of $(\Sigma, \mathcal{O})$. The countability of $\mathcal{B}$ follows from that of $\Sigma \setminus \Sigma_4$ and $\mathcal{N}(s)$. □

We obtain the following lemma by Lemmas 3.40 and 3.41.

Lemma 3.42. *The topological space $(\Sigma, \mathcal{O})$ is compact.*

By Theorem 3.32, $(\Sigma, \mathcal{O})$ is normal, in particular $(\Sigma, \mathcal{O})$ is regular. Therefore we obtain the following lemma by Theorem 3.33.

Lemma 3.43. *The topological space $(\Sigma, \mathcal{O})$ is metrizable.*

Next, we show that $(\Sigma, \mathcal{O})$ is totally disconnected. By Theorem 3.35, we need only to show that $(\Sigma, \mathcal{O})$ is a 0-dimensional space.

Lemma 3.44. *Let $s \in \Sigma \setminus \Sigma_4$.*

- (i) *If $s = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ or $s = (\mathsf{B}, \mathsf{B}, \mathsf{B}, \dots)$, then $N_s^{(k)}$ is open and closed for $k \geq 0$.*
- (ii) *If $s = (s_0, \dots, s_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ with $s_l \neq \mathsf{A}$ or $s = (s_0, \dots, s_l, \mathsf{B}, \mathsf{B}, \mathsf{B}, \dots)$ with $s_l \neq \mathsf{B}$, then $N_s^{(k)}$ is open and closed for $k \geq l + 1$.*

Proof. (i) Let $s = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$. We show that $N_s^{(k)}$ is closed. Let $M = \Sigma \setminus N_s^{(k)}$. For a point $t = (t_n)$ in M, there exists $\alpha \leq k$ such that $t_\alpha \neq 1$ or 2. In the case of "B", there exists $\alpha \leq k$ such that $t_\alpha \neq 3$ or 4. Then $N_s^{(k)} \cap N_t^{(\alpha)} = \emptyset$ and $N_t^{(\alpha)} \subset M$. Therefore M is open and $N_s^{(k)}$ is closed. The proof of (ii) is similar to that of (i). □

For $s = (\mathsf{A}, \mathsf{A}, \mathsf{A} \dots)$ or $s = (\mathsf{B}, \mathsf{B}, \mathsf{B} \dots)$, we set $\mathcal{N}'(s) = \mathcal{N}(s)$. For $s = (s_0, \dots, s_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ with $s_l \neq \mathsf{A}$ or $s = (s_0, \dots, s_l, \mathsf{B}, \mathsf{B}, \mathsf{B}, \dots)$ with $s_l \neq \mathsf{B}$, we set $\mathcal{N}'(s) = \{N_s^{(k)} : k \geq l + 1\}$. Let

$$\mathcal{B}' = \bigcup_{s \in \Sigma \setminus \Sigma_4} \mathcal{N}'(s).$$

Lemma 3.45. *The set $\mathcal{B}'$ is an open basis of* $(\Sigma, \mathcal{O})$.

Proof. The proof is smiler to that of Lemma 3.41. □

By Lemmas 3.44 and 3.45, $(\Sigma, \mathcal{O})$ is a 0-dimensional space. Therefore we obtain the following lemma by Theorem 3.35.

Lemma 3.46. *The topological space* $(\Sigma, \mathcal{O})$ *is totally disconnected.*

Finally, we show the following lemma.

Lemma 3.47. *The shift map* $\sigma : (\Sigma, \mathcal{O}) \to (\Sigma, \mathcal{O})$ *is continuous.*

Proof. Let $s = (s_0, s_1, s_2, \dots) \in \Sigma$. For a neighborhood $O \in \mathcal{O}$ of $\sigma(s) = (s_1, s_2, \dots)$, there exists $N = N^{(k)}_{\sigma(s)} \in \mathcal{N}(\sigma(s))$ such that $N \subset O$. We take a neighborhood $M = N^{(k+1)}_s$ of s. Then $\sigma(M) = N \subset O$. Therefore $\sigma : (\Sigma, \mathcal{O}) \to (\Sigma, \mathcal{O})$ is continuous. □

We have completed the proof of Theorem 3.23.

Applications.
The following two theorems are fundamental. See Section **??**.

Theorem 3.48. *Let g be a rational function of degree greater than one. If z belongs to $J(g)$, then*

$$J(g) = \overline{\bigcup_{k=1}^{\infty} g^{-k}(z)}.$$

Theorem 3.49. *Let g be a rational function of degree greater than one. Then*

$$J(g) = \overline{\{\text{repelling periodic point of } g\}}.$$

We obtain analogies of Theorems 3.48 and 3.49.

Theorem 3.50. *Let* $(\Sigma, \mathcal{O})$ *be as in Theorem* 3.23 *and let s be a point in Σ. Then*

$$\Sigma = \overline{\bigcup_{k=1}^{\infty} \sigma^{-k}(s)},$$

where the closure is taken in $(\Sigma, \mathcal{O})$.

Proof. Let $s = (s_0, s_1, s_2, \dots) \in \Sigma$ and let

$$u = \begin{cases} 1 \text{ or } 2 & \text{if } s_0 \neq \mathsf{A}, \\ 3 \text{ or } 4 & \text{if } s_0 \neq \mathsf{B}. \end{cases}$$

Then $(u, s_0, s_1, s_2, \dots) \in \sigma^{-1}(s)$. For a point $t = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ in Σ, we consider the sequence

$$\left\{ s^{(\alpha)} = (\underbrace{1, \dots, 1}_{\alpha\text{-times}}, u, s_0, s_1, \dots) \right\}_{\alpha=1}^{\infty} \subset \bigcup_{k=1}^{\infty} \sigma^{-k}(s).$$

Then $s^{(\alpha)}$ converges to $t = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ with respect to $\mathcal{O}$. Next, for a point $t = (t_0, t_1, \dots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ in Σ with $t_l \neq \mathsf{A}$, we consider the sequence

$$\left\{ s^{(\alpha)} = (t_0, \dots, t_l, \underbrace{1, \dots, 1}_{\alpha\text{-times}}, u, s_0, s_1, \dots) \right\}_{\alpha=1}^{\infty} \subset \bigcup_{k=1}^{\infty} \sigma^{-k}(s).$$

Then $s^{(\alpha)}$ converges to $t = (t_0, t_1, \dots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ with respect to $\mathcal{O}$. In the case of "B", we choose "3" instead of "1". Finally, for a point $t = (t_0, t_1, t_2, \dots)$ in $\Sigma \cap \Sigma_4$, we consider the sequence

$$\left\{ s^{(\alpha)} = (t_0, t_1, \dots, t_\alpha, u, s_0, s_1, s_2, \dots) \right\}_{\alpha=1}^{\infty} \subset \bigcup_{k=1}^{\infty} \sigma^{-k}(s).$$

Then $s^{(\alpha)}$ converges to $t = (t_0, t_1, t_2, \dots)$ with respect to $\mathcal{O}$. □

Remark 3.51. The closure of the backward orbit of $s \in \Sigma$ under σ does not necessarily coincide with Σ in (Σ, ρ). For example,

$$(\mathsf{A}, \mathsf{A}, \mathsf{A}, \dots) \notin \overline{\bigcup_{\mathsf{k}=1}^{\infty} \sigma^{-\mathsf{k}}((\mathsf{B}, \mathsf{B}, \mathsf{B}, \dots))},$$

where the closure is taken in (Σ, ρ).

Corollary 3.52. *Let $(K(f)^*, \mathcal{G})$ be as in Corollary* 3.24 *and let $K \in K(f)^*$. Then*

$$K(f)^* = \overline{\bigcup_{k=1}^{\infty} F^{-k}(K)},$$

where the closure is taken in $(K(f)^*, \mathcal{G})$.

Theorem 3.53. *Let $(\Sigma, \mathcal{O})$ be as in Theorem* 3.23. *Then*

$$\Sigma = \overline{\{\text{periodic point of } \sigma \text{ in } \Sigma\}},$$

where the closure is taken in $(\Sigma, \mathcal{O})$.

Proof. We show that each non-periodic point t in Σ is a limit point of a sequence of periodic points of Σ. For a point $t = (t_0, t_1, \dots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ in Σ with $t_l \neq \mathsf{A}$, we consider the sequence

$$\left\{ s^{(\alpha)} = (t_0, t_1, \dots, t_l, \underbrace{1, 1, \dots, 1}_{\alpha\text{-times}},\ t_0, t_1, \dots, t_l, \underbrace{1, 1, \dots, 1}_{\alpha\text{-times}}, \dots) \right\}_{\alpha=1}^{\infty}$$

of period $\alpha + l + 1$. Then $s^{(\alpha)}$ converges to $t = (t_0, t_1, \dots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ with respect to $\mathcal{O}$. In the case of "B", we choose "3" instead of "1". For a non-periodic point $t = (t_0, t_1, t_2, \dots)$ in $\Sigma \cap \Sigma_4$, we consider the sequence

$$\left\{ s^{(\alpha)} = (t_0, t_1, \dots, t_\alpha, t_0, t_1, \dots, t_\alpha, \dots) \right\}_{\alpha=\beta}^{\infty}$$

of period $\alpha + 1$, where β is a positive integer which satisfies $s^{(\beta)} \in \Sigma$. Then $s^{(\alpha)}$ converges to $t = (t_0, t_1, t_2, \dots)$ with respect to $\mathcal{O}$. □

Remark 3.54. The closure of the set of all periodic points of Σ does not coincide with Σ in (Σ, ρ) since $t = (t_0, t_1, \dots, t_l, \mathsf{A}, \mathsf{A}, \mathsf{A}, \dots)$ with $t_l \neq \mathsf{A}$ is an isolated point in (Σ, ρ).

Corollary 3.55. *Let* $(K(f)^*, \mathcal{G})$ *be as in Corollary* 3.24. *Then*

$$K(f)^* = \overline{\{\text{periodic point of } F \text{ in } K(f)^*\}},$$

where the closure is taken in $(K(f)^*, \mathcal{G})$.

References

[1] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Second edition, With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. University Lecture Series, 38. American Mathematical Society, Providence, RI, 2006.

[2] A. Avila, X. Buff and A. Chéritat, Siegel disks with smooth boundaries. *Acta Math.* **193** (2004), no. 1, 1–30.

[3] A. F. Beardon, Symmetries of Julia sets, *Bull. London Math. Soc.* **22** (1990), no. 6, 576–582.

[4] A. F. Beardon, *Iteration of Rational Functions*, Complex analytic dynamical systems, Graduate Texts in Mathematics, 132. Springer-Verlag, New York, 1991.

[5] W. Bergweiler, On the number of critical points in parabolic basins, *Ergodic Theory Dynam. Systems* **22** (2002), no. 3, 655–669.

[6] P. Bhattacharyya and Y. E. Arumaraj, On the iteration of polynomials of degree 4 with real coefficients, *Ann. Acad. Sci. Fenn. Math.* **6** (1981), no. 2, 197–203.

[7] P. Bhattacharyya and M. Tamil Durai, On the Julia set of polynomials with complex coefficients, *J. Math. Phys. Sci.* **29** (1995), no. 3, 119–129.

[8] P. Blanchard, Complex analytic dynamics on the Riemann sphere, *Bull. Amer. Math. Soc.* (*N.S.*) **11** (1984), no. 1, 85–141.

[9] P. Blanchard, Disconnected Julia sets, *Chaotic Dynamics and Fractals*, Academic Press, 1986.

[10] P. Blanchard, R. L. Devaney, D. M. Look, P. Seal and Y. Shapiro, Sierpinski-curve Julia sets and singular perturbations of complex polynomials, *Ergodic Theory Dynam. Systems* **25** (2005), no. 4, 1047–1055.

[11] X. Buff and A. Chéritat, Quadratic Julia sets with positive area, `http://www.picard.ups-tlse.fr/~buff/Preprints/Preprints.html`.

[12] X. Buff and C. Henriksen, Scaling ratios and triangles in Siegel disks, *Math. Res. Lett.* **6** (1999), no. 3-4, 293–305.

[13] L. Carleson and T. Gamelin, *Complex Dynamics*, Springer-Verlag, 1993.

[14] A. Douady, Does a Julia set depend continuously on the polynomial?, *Complex dynamical systems* (*Cincinnati, OH, 1994*), 91–138, Proc. Sympos. Appl. Math., 49, Amer. Math. Soc., Providence, RI, 1994.

[15] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like mappings, *Ann. Sci. École Norm. Sup.* (*4*) **18** (1985), no. 2, 287–343.

[16] A. L. Epstein, Bounded hyperbolic components of quadratic rational maps, *Ergodic Theory Dynam. Systems* **20** (2000), no. 3, 727–748.

[17] N. Fagella and L. Geyer, Surgery on Herman rings of the complex standard family, *Ergodic Theory Dynam. Systems* **23** (2003), no. 2, 493–508.

[18] A. Fletcher and V. Markovic, *Quasiconformal Maps and Teichmüller Theory*, Oxford Graduate Texts in Mathematics 11, Oxford University Press, Oxford, 2007.

[19] L. Geyer, Siegel discs, Herman rings and the Arnold family, *Trans. Amer. Math. Soc.* **353** (2001), no. 9, 3661–3683.

[20] J. Graczyk and P. Jones, Dimension of the boundary of quasiconformal Siegel disks, *Invent. Math.* **148** (2002), no. 3, 465–493.

[21] A. Hinkkanen and G. J. Martin, Julia sets of rational semigroups, *Math. Z.* **222** (1996), no. 2, 161–169.

[22] A. Hinkkanen and G. J. Martin, The dynamics of semigroups of rational functions I, *Proc. London Math. Soc.* (*3*) **73** (1996), no. 2, 358–384.

[23] K. Katagata, Dynamics of even polynomials with real coefficients, *Int. Journal of Math. Analysis*, **2** (2008), no 2, 85–96.

[24] K. Katagata, Some cubic Blaschke products and quadratic rational functions with Siegel disks, *Int. J. Contemp. Math. Sci.* **2** (2007), no. 30, 1455–1470.

[25] K. Katagata, Blaschke products and rational functions with Siegel disks, *J. Korean Math. Soc.*, to appear.

[26] K. Katagata, On a certain kind of polynomials of degree 4 with disconnected Julia set, *Discrete Contin. Dyn. Syst.* **20** (2008), no. 4, 975–987.

[27] K. Katagata, Disconnected Julia sets of quartic polynomials and a new topology of the symbol space, *Proc. Japan Acad. Ser. A Math. Sci.* **84** (2008), no. 7, 117–122.

[28] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications 54, Cambridge University Press, 1995.

[29] J. L. Kelley, *General Topology*, Springer, 1975.

[30] K. Linda and Z. Gaofei, Bounded type Siegel disks of a one dimensional family of entire functions, preprint.

[31] J. Lü and W. Qiu, Connectivity of Julia sets for even quartic polynomials, *Chinese J. Contemp. Math.* **24** (2003), no. 3, 197–204.

[32] C. T. McMullen, Self-similarity of Siegel disks and Hausdorff dimension of Julia sets, *Acta Math.* **180** (1998), no. 2, 247–292.

[33] W. de Melo and S. van Strien, *One-dimensional Dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 25, Springer-Verlag, Berlin, 1993.

[34] J. Milnor, *Dynamics in One Complex Variable*, Vieweg, 2nd edition, 2000.

[35] J. Milnor, Geometry and dynamics of quadratic rational maps, With an appendix by the author and Lei Tan, *Experiment. Math.* **2** (1993), no. 1, 37–83.

[36] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, *Holomorphic Dynamics*, Cambridge Studies in Advanced Mathematics 66, 2000.

[37] J. Nagata, *Modern General Topology*, American Elsevier Publishing Company, 2nd revised edition, 1974.

[38] C. L. Petersen, On holomorphic critical quasi-circle maps, *Ergodic Theory Dynam. Systems* **24** (2004), no. 5, 1739–1751.

[39] N. Steinmetz, *Rational Iteration*, Complex analytic dynamical systems, de Gruyter Studies in Mathematics, 16, Walter de Gruyter & Co., Berlin, 1993.

[40] H. Sumi, On dynamics of hyperbolic rational semigroups, *J. Math. Kyoto. Univ.* **37** (1997), no. 4, 717–733.

[41] H. Sumi, On Hausdorff dimension of Julia sets of hyperbolic rational semigroups. *Kodai Math. J.* **21** (1998), no. 1, 10–28.

[42] H. Sumi, Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products, *Ergodic Theory Dynam. Systems* **21** (2001), no. 2, 563–603.

[43] M. Yampolsky and S. Zakeri, Mating Siegel quadratic polynomials, *J. Amer. Math. Soc.* **14** (2001), no. 1, 25–78 (electronic).

[44] S. Zakeri, Dynamics of cubic Siegel polynomials, *Comm. Math. Phys.* **206** (1999), no. 1, 185–233.

[45] S. Zakeri, Old and new on quadratic Siegel disks, `http://www.math.qc.edu/~zakeri/papers/papers.html`.

Koh Katagata: Interdisciplinary Faculty of Science and Engineering, Shimane University, Matsue, Shimane, 690-0822, Japan

E-mail address: `katagata@math.shimane-u.ac.jp`