Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
34 (2001), pp.87–104

SPECTRAL CURVES, DRESSING ACTIONS AND HARMONIC MAPS OF R^2 INTO COMPLEX GRASSMANN MANIFOLDS

SEIICHI UDAGAWA

Communicated by S. Maeda

(Received: December 24, 2000)

INTRODUCTION

In 1993, Burstall-Ferus-Pedit-Pinkall([BFPP]) found a certain class of solutions to the harmonic map equations of $\mathbf{R}^2$ into compact symmetric space and they called a harmonic map which belongs to this class a harmonic map *of finite type*. In case of two-torus, a sufficient condition for a harmonic map into compact symmetric space be of finite type is semisimplicity of the $(1,0)$-part of the pullback of the Maurer-Cartan form for target symmetric space, which is denoted by $\alpha'_{\mathfrak{m}}$. In this case, the corresponding harmonic map is said to be of *semisimple* finite type. These results and notions were extended as *primitive* harmonic map of semisimple finite type for k-symmetric spaces as target in [BP] and [B]. Using this sufficient condition, they showed that any non-conformal harmonic map of two-torus into compact symmetric space of rank one is of finite type. After their works, it was shown that some harmonic maps of two-torus are lifted to primitive harmonic maps of semisimple finite type into some k-symmetric space. For examples, non-isotropic weakly conformal harmonic two-torus essentially has such a characterization when the target manifold is

(1) a sphere or a complex projective space([B]),
(2) $G_2(\mathbf{C}^4)$ or $\mathbf{H}P^3$([U1], [U2]),

where $G_2(\mathbf{C}^4)$ is a complex Grassmannian manifold of 2-planes in $\mathbf{C}^4$ and $\mathbf{H}P^3$ is a 3-dimensional quaternionic projective space. After a while, Burstall and Pedit ([BP2]) showed that any primitive harmonic map of semisimple finite type of $\mathbf{R}^2$ into a k-symmetric space is in a dressing orbit of a vacuum solution, where vacuum solution is a primitive harmonic map with $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ being a constant normal matrix for fixed coordinate z on $\mathbf{R}^2$, hence of semisimple finite type. After that, McIntosh([M2], [M3]) gave another method of constructing non-isotropic harmonic maps (which is covered by primitive harmonic map of semisimple finite type)of $\mathbf{R}^2$ into a complex projective space. Moreover, he showed that there is a bijective correspondence between the spectral data $\{\pi : X \to \mathbf{P}^1, \mathcal{L}\}$ which satisfies the certain conditions and the linearly full non-isotropic harmonic maps of $\mathbf{R}^2$ into a complex projective space, which are of finite type by a recent result obtained in [OU].

In this paper, we review the McIntosh's construction of harmonic maps into complex projective space and give some examples coming from some spectral curves of genus zero and one. Moreover, we give some examples of harmonic maps of $\mathbf{R}^2$ into $G_2(\mathbf{C}^4)$ extending the method of McIntosh's construction. They are harmonic maps of *non-semisimple* finite type. When the pullback of the Maurer-Cartan form $\alpha'(\partial/\partial z)$ is a constant matrix for fixed coordinate z, the corresponding harmonic map is $\mathbf{R}^2$-equivariant and of finite type. If $\alpha'(\partial/\partial z)$ is a constant matrix with respect to some fixed coordinate system, then it is a semisimple matrix, however, $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ is not always semisimple. When $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ is non-semisimple matrix, there are essentially two types of α' after some choices of the basis and their S^1-families. One of them is obtained by spectral curve of genus zero and some hyperplane line bundle of degree 3. The other one is obtained by dressing action of the former one. It is observed that the dressing actions preserves the isotropy order of the harmonic map. The harmonic map obtained by the spectral curves of higher genus is in the dressing orbit of the harmonic map with same isotropy order obtained by the spectral curves of genus zero. In fact, the relation between the dressing actions on harmonic maps of finite type and the dressing actions on the spectral data is made clear.

1. Review of McIntosh's theory

First of all, we give a definition of the spectral data :

Definition. A triple $(X, \pi, \mathcal{L})$ is said to be a *spectral data* if they satisfy the following conditions ;

(1) X is a compact Riemann surface of genus p with real structure (i.e., anti-holomorphic involution) ρ_X,

(2) there is a holomorphic covering map $\pi : X \longrightarrow \mathbf{P}^1$ with $\deg(\pi) = n+1$ such that $\pi \circ \rho_X = \overline{\pi}^{-1}$(=a real structure of $\mathbf{P}^1$) and the divisor of π is given by

$$(\pi) = (m+1)P_0 + P_1 + \cdots P_{n-m} - (m+1)Q_0 - Q_1 - \cdots - Q_{n-m},$$

where $Q_j = \rho_X(P_j)$ for $j = 0, 1, \cdots, n-m$,

(3) $\mathcal{L}$ is a complex line bundle over X of degree $(n+p)$ such that $f : \mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} \longrightarrow \mathcal{O}_X(R)$ is a ρ_X-equivariant ($\overline{\rho_{X*}f} = f$) isomorphism, where R is the ramification divisor of π and $\mathcal{O}_X(R)$ is the line bundle corresponding to R,

(4) ρ_X fixes each point of the preimage $X_{\mathbf{R}}$ of th equator S^1_λ of $\mathbf{P}^1_\lambda$, π has no branch points on $X_{\mathbf{R}}$ and f is non-negative on $X_{\mathbf{R}}$.

Given a spectral data $(X, \pi, \mathcal{L})$, we see from the Riemann-Hurwitz formula that $\deg(R) = 2n + 2p$. Then, the condition (4) above guarantees that there is some positive divisor D on X with $\deg(D) = n + p$ such that $R = D + \rho_{X*}(D)$. If we identify $\mathcal{L}$ with a divisor line bundle $\mathcal{O}_X(D_0)$ for some divisor D_0 on X, the condition (3) above implies that f is a rational function on X with a divisor (f) given by $(f) = (D_0 + \rho_{X*}(D_0) - D - \rho_{X*}(D))$. We introduce a Hermitian inner product on $H^0(X, \mathcal{L})$, which is the vector space of all global holomorphic sections of $\mathcal{L}$. Denote by $\pi_*\mathcal{L}$ the direct image sheaf of $\mathcal{L}$ by π, i.e., $\Gamma(U, \pi_*\mathcal{L}) = \Gamma(\pi^{-1}(U), \mathcal{L})$ for arbitrary open subset U of $\mathbf{P}^1_\lambda$. Therefore, $H^0(X, \mathcal{L}) = H^0(\mathbf{P}^1_\lambda, \pi_*\mathcal{L})$. Set $A = \mathbf{P}^1_\lambda \setminus \{0, \infty\}$ and $I = I_0 \cup I_\infty$, where I_0 (resp. I_∞) is an open neighborhood around $\lambda = 0$ (resp. $\lambda = \infty$) which contains no branch points except $\lambda = 0$ (resp. except $\lambda = \infty$). Hence, $\mathbf{P}^1_\lambda = A \cup I$. Next, set $X_A = \pi^{-1}(A)$ and $X_I = \pi^{-1}(I)$ so that $X = X_A \cup X_I$. Then we define a bilinear form h on $\Gamma(X_A, \mathcal{L}) \times \Gamma(X_A, \mathcal{L})$ by

$$h : \Gamma(X_A, \mathcal{L}) \times \Gamma(X_A, \mathcal{L}) \ni (v, w) \to \mathrm{Tr}(f \cdot v \otimes \overline{\rho_{X*} w}) \in \mathbf{C}[\lambda^{-1}, \lambda],$$

where $\mathbf{C}[\lambda^{-1}, \lambda]$ is the ring generated by λ, λ^{-1} over the field $\mathbf{C}$. Notice that $f \cdot v \otimes \overline{\rho_{X*} w}$ is a holomorphic section of $\mathcal{O}_X(R)$ over X_A, whence its trace is a holomorphic function on A. Take a point P of $\mathbf{P}^1_\lambda$. Then we have

$$h(v, w)(P) = \sum_{x \in \pi^{-1}(P)} f(x) \cdot v(x) \overline{w(\rho_X(x))}.$$

The summation is taken over all points $\{x_0, \cdots, x_n\} = \pi^{-1}(P)$ and it is counted with multiplicities if P is a (or an image of) branch point. Since a global holomorphic function on $\mathbf{P}^1_\lambda$ is a constant and f is ρ_X-equivariant, we see that $h\,|_{H^0 \times H^0}$ defines a Hermitian symmetric form. The positive definiteness of $h\,|_{H^0 \times H^0}$ depends on the choice of f. The condition (4) of the spectral data guarantees that $h\,|_{H^0 \times H^0}$ is positive definite. Thus, $h\,|_{H^0 \times H^0}$ is a Hermitian inner product.

Lemma 1.1([M2]). *$\pi_*\mathcal{L}$ is a rank $(n+1)$ trivial bundle over $\mathbf{P}^1_\lambda$.*

Since $h^0(X, \mathcal{L}) = \dim H^0(X, \mathcal{L}) = n + 1$, it follows from the Riemann-Roch formula that $h^1(X, \mathcal{L}) = 0$, in which case $\mathcal{L}$ is called *non-special.*

To construct a map from $\mathbf{R}^2$, we need to define a parallel transport of a section of $\mathcal{L}$ to a section of a line bundle over $\mathbf{R}^2$. Let $J(X)$ be the Jacobian variety of the spectral curve X. The set of all line bundles $L \in J(X)$ which satisfy $\overline{\rho_{X*} L} \cong L^{-1}$ forms a subgroup of $J(X)$ by a tensor product. We denote by $J_{\mathbf{R}}(X)$ the connected component of the identity of this subgroup. For any $L \in J_{\mathbf{R}}(X)$, we see that a line bundle $\mathcal{L} \otimes L$ satisfies $(\mathcal{L} \otimes L) \otimes \overline{\rho_{X*}(\mathcal{L} \otimes L)} \cong \mathcal{O}_X(R)$. In this case, we say that $\mathcal{L} \otimes L$ is *real.* Note that when we replace $\mathcal{L}$ by $\mathcal{L} \otimes L$ for $L \in J_{\mathbf{R}}(X)$ we see that f is still non-negative on the preimage $X_{\mathbf{R}}$ of the equator S^1_λ. Since $\deg(\mathcal{L} \otimes L) = n + p$, it follows from Lemma 1.1 that $\pi_*(\mathcal{L} \otimes L)$ is a rank $(n+1)$ trivial bundle and $h^0(X, \mathcal{L} \otimes L) = n + 1$. Now, consider a complex vector bundle $H^0(X) \to J_{\mathbf{R}}(X)$ of which the fibre at $L \in J_{\mathbf{R}}(X)$ is given by a $(n+1)$-dimensional complex vector space $H^0(X, \mathcal{L} \otimes L)$. Recall that $X = X_A \cup X_I$. A line bundle $L \in J(X)$ is trivialized over X_A or X_I. We denote by θ_A and θ_I trivializing sections over X_A and X_I, respectively. Over $X_A \cap X_I$, we have a transition relation $\theta_I = e^a \theta_A$. Thus, for $L \in J_{\mathbf{R}}(X)$, we have a 1-cocycle (e^a, X_A, X_I). Conversely, a 1-cocycle (e^a, X_A, X_I) defines a line bundle L with e^a as a transition function. Then, consider a map $L : \mathfrak{g} = \Gamma(X_A \cap X_I, \mathcal{O}_X) \longrightarrow J(X)$ defined by $a \to L(a)$, where $L(a)$ is a line bundle with a transition function e^a. Set $\mathfrak{g}_{\mathbf{R}} = \{a \in \mathfrak{g} \mid \overline{\rho_{X*} a} = -a\}$. Then, we see that $\mathrm{Im}(L \mid_{\mathfrak{g}_{\mathbf{R}}}) = J_{\mathbf{R}}(X)$. Now, fix a trivializing section θ for $\mathcal{L}$ over X_I such that $\mathrm{Tr}(f \cdot \theta \otimes \overline{\rho_{X*}\theta}) = 1$. For $a \in \mathfrak{g}_{\mathbf{R}}$, set $\theta_a = \theta \otimes \theta_I$, which is a trivializing section for $\mathcal{L} \otimes L(a)$ over X_I. For $\sigma_a \in \Gamma(X_A, \mathcal{L} \otimes L(a))$, define $\iota_a(\sigma_a)$ by $\iota_a(\sigma_a) = e^a(\sigma_a \theta_a^{-1})\theta$. Then, we have $\iota_a(\sigma_a) \in \Gamma(X_A, \mathcal{L})$(see [M2]).

Consider a map $a : \mathbf{R}^2 \longrightarrow \mathfrak{g}_{\mathbf{R}}$ defined by $z \to a(z, \bar{z}) = z\zeta^{-1} - \bar{z}\zeta$, where ζ is considered only on $X_A \cap (U_0 \cup U_\infty)$, where U_0 (resp. U_∞) is a connected component of X_0 (resp. X_∞) which contains P_0 (resp. Q_0). Then $L(a)$ is a 2-parameter subgroup of $J_{\mathbf{R}}(X)$. Fix h-orthonormal basis $\{\tau_j\}$ for $H^0(X, \mathcal{L})$ such that $(H^0(X, \mathcal{L}), h) \longrightarrow (\mathbf{C}^{n+1}, <\ ,\ >)$ is isometric. We decompose the vector bundle $H^0(X) \to \mathbf{R}^2$ into line subbundles which are orthogonal to each other. For the purpose, define the following line bundles, of which the sheaf of germs of holomorphic sections are subsheaves of the sheaf of germs of holomorphic sections of $\mathcal{L}$:

(1.1)

$$\begin{cases} \mathcal{L}_j = \mathcal{L} \otimes \mathcal{O}_X(-(m-j)P_0 - jQ_0 - \displaystyle\sum_{i=1}^{n-m} P_i) & \text{for} \quad j = 0, 1, \cdots, m-1, \\ \mathcal{L}_m = \mathcal{L} \otimes \mathcal{O}_X(-mQ_0). \end{cases}$$

Then, each $\mathcal{L}_j$ is non-special for $j = 0, 1, \cdots, m$. Therefore, we also see that each $\mathcal{L}_j \otimes L(a)$ is non-special for $j = 0, 1, \cdots, m$ and for arbitrary $a \in \mathfrak{g}_{\mathbf{R}}$. The Riemann-Roch theorem yields that

$$h^0(X, \mathcal{L}_j \otimes L(a)) = \begin{cases} 1 & \text{for} \quad j = 0, 1, \cdots, m-1, \\ n+1-m & \text{for} \quad j = m. \end{cases}$$

We have

$$H^0(X, \mathcal{L} \otimes L(a)) = \bigoplus_{j=0}^{m} H^0(X, \mathcal{L}_j \otimes L(a)) \quad (h-\text{orthogonal sum}).$$

Let $\{s_0, \cdots, s_n\}$ be an orthonormal basis for $H^0(X, \mathcal{L})$ such that $s_j \in H^0(X, \mathcal{L}_j)$ for $j = 0, 1, \cdots, m-1$ and $s_m, \cdots, s_n \in H^0(X, \mathcal{L}_m)$. Set $s_i(z) = \iota_a^{-1}(s_i)$. Then $\{s_0(z), \cdots, s_n(z)\}$ is an orthonormal basis for $H^0(X, \mathcal{L} \otimes L(a))$ such that $s_j(z) \in$

$H^0(X, \mathcal{L}_j \otimes L(a))$ for $j = 0, 1, \cdots m-1$ and $s_m(z), \cdots, s_n(z) \in H^0(X, \mathcal{L}_m \otimes L(a))$ (see [M2]). Define $F(z, \lambda)$, which depends only on z and λ by

$$(s_0(z), \cdots, s_n(z)) = (s_0, \cdots, s_n)F(z, \lambda). \tag{1.2}$$

Let $f(\zeta)$ be arbitrary regular algebraic function of ζ on X_A. Define $Y(z, \lambda)$ by

$$(s_0(z), \cdots, s_n(z))Y(z, \lambda) = f(\zeta)(s_0(z), \cdots, s_n(z)).$$

Then we obtain $Y(z, \lambda) = \mathrm{Ad}F(z, \lambda)^{-1} \cdot Y(0, \lambda)$ and $dY = [Y, F^{-1}dF]$, i.e., $Y(z, \lambda)$ is a polynomial Killing field. Thus, the corresponding map is a primitive harmonic map of finite type and a non-isotropic harmonic map $\varphi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^n$ is given by the first column vector of $F(z, 1)$. To obtain an explicit form of the non-isotropic harmonic map φ, we may use (1.2). Let $\{\zeta_0, \cdots, \zeta_n\}$ be the element of $\pi^{-1}(1)$, which are different from each other by the condition of the spectral data (4). We evaluate the equation (1.2) at $\zeta = \zeta_0, \cdots, \zeta_n$. Set

$$M(z) = \begin{pmatrix} s_0(z) \mid_{\zeta_0} & \cdots & s_n(z) \mid_{\zeta_0} \\ \vdots & \ddots & \vdots \\ s_0(z) \mid_{\zeta_n} & \cdots & s_n(z) \mid_{\zeta_n} \end{pmatrix} \tag{1.3}$$

Then, (1.3), together with (1.2), yields that $M(z) = M(0)F(z, 1)$. Now, we must show that $M(0)$ is non-singular. Let $S_0, \cdots, S_n$ be a local frame around $\pi^{-1}(1)$ which has the properties $S_i(\zeta_j) = 0$ for $i \neq j$ and $f \cdot S_i(\zeta_i)\overline{\rho_{X*}S_i(\zeta_i)} = 1$. If we express s_i around $\pi^{-1}(1)$ as $s_i = \sum_{j=0}^n v_{ij}S_j$, then we see that

$$\begin{aligned} \delta_{ij} &= h(s_i, s_k) \\ &= \sum_{j=0}^{n} (f \cdot \sum_l v_{il} S_l(\zeta_j) \overline{\sum_m v_{km} S_m(\rho_{X*}\zeta_j)} \\ &= \sum_{j=0}^{n} v_{ij}\overline{v_{kj}} \quad , \end{aligned}$$

which shows that $M(0)$ is non-singular. Therefore, we have $F(z, 1) = M(0)^{-1}M(z)$.

2. Some examples of harmonic maps of $\mathbf{R}^2$ into $\mathbf{C}P^n$

In this section, we give some examples of harmonic maps of $\mathbf{R}^2$ into complex projective space via McIntosh's construction.

Example 2.1. Consider $\pi : \mathbf{C}P^1_\zeta \longrightarrow \mathbf{C}P^1_\lambda$ defined by $\zeta \mapsto \lambda = \zeta^{-3}$. Then $(\pi) = 3(\infty) - 3(0)$ and $R = 2(\infty) + 2(0)$, we must regard $P_0 = \{\zeta = \infty\}$, $Q_0 = \{\zeta = 0\}$. Define $\mathcal{L}$ by $\mathcal{L} = \mathcal{O}_X(2(\infty))$. Then we have

$$\begin{cases} \mathcal{L}_0 = \mathcal{L} \otimes \mathcal{O}_X(-2P_0) = \mathcal{O}_X, \\ \mathcal{L}_1 = \mathcal{L} \otimes \mathcal{O}_X(-P_0 - Q_0) = \mathcal{O}_X((\infty) - (0)), \\ \mathcal{L}_2 = \mathcal{L} \otimes \mathcal{O}_X(-2Q_0) = \mathcal{O}_X(2(\infty) - 2(0)), \end{cases}$$

which implies that we may set $s_0 = 1, s_1 = \zeta, s_2 = \zeta^2$ and $s_0(z) = e^a, s_1(z) = e^a\zeta, s_2(z) = e^a\zeta^2$, where $a = z\zeta - \overline{z}\zeta^{-1}$. Since $\pi^{-1}(1) = \{1, \omega, \omega^2\}$, where $\omega = (-1 - \sqrt{-3})/2$, we have

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad M(z) = \begin{pmatrix} e^{z-\overline{z}} & e^{z-\overline{z}} & e^{z-\overline{z}} \\ e^{z\omega - \overline{z\omega}} & e^{z\omega - \overline{z\omega}}\omega & e^{z\omega - \overline{z\omega}}\omega^2 \\ e^{z\omega^2 - \overline{z\omega^2}} & e^{z\omega^2 - \overline{z\omega^2}}\omega^2 & e^{z\omega^2 - \overline{z\omega^2}}\omega \end{pmatrix}.$$

Therefore, we can easily obtain $F(z,1) = M^{-1} \cdot M(z)$ and we see that $\varphi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^2$ is a harmonic map of isotropy order 2. Note that φ is doubly periodic. We recommend the readers to obtain the explicit form of $F(z,1)$. In the same way, we may calculate $F(z,1)$ in the case where π is given by $\pi(\zeta) = \zeta^{-(n+1)}$, where we choose $\mathcal{L} = \mathcal{O}_X(n(\infty))$ and obtain a harmonic map $\varphi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^n$ of isotropy order n (cf. [T]), which is doubly periodic for $n = 1, 2, 3, 5$. The case of $\pi(\zeta) = \zeta^{n+1}$ is also similar, where we choose $\mathcal{L} = \mathcal{O}_X(n(0))$ and we see that $1, \zeta^{-1}, \cdots, \zeta^{-n}$ are global sections, respectively, of $\mathcal{L}_0, \mathcal{L}_1, \cdots, \mathcal{L}_n$.

Example 2.2. Consider $\pi : X \longrightarrow \mathbf{P}^1_\lambda$ defined by $\pi(\zeta) = \frac{1}{\alpha}\zeta^3 \frac{(\zeta-\alpha)}{(\zeta-\alpha^{-1})} = \lambda$, where $0 < \alpha < 1$. Then we have $(\pi) = 3(0) + (\alpha) - 3(\infty) - (\alpha^{-1})$, $(R) = 2(0) + (p) + (p^{-1}) + 2(\infty)$ and $X = \mathbf{P}^1_\zeta$, where

$$p = \frac{\alpha^2 + 2 - \sqrt{\alpha^4 - 5\alpha^2 + 4}}{3\alpha}.$$

Hence, there is no branch points on $X_{\mathbf{R}}$. In this case, $P_0 = \{\zeta = 0\}, P_1 = \{\zeta = \alpha\}, Q_0 = \{\zeta = \infty\}$ and $Q_1 = \{\zeta = \alpha^{-1}\}$. Define $\mathcal{L} = \mathcal{O}_X(3P_0)$ Then $f : \mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} \longrightarrow \mathcal{O}_X(R)$ is given by

$$f = \frac{-\zeta}{(\zeta - p)(\zeta - p^{-1})},$$

which is non-negative on $X_{\mathbf{R}}$. We have

$$\begin{cases} \mathcal{L}_0 = \mathcal{L} \otimes \mathcal{O}_X(-2P_0 - P_1) = \mathcal{O}_X(P_0 - P_1), \\ \mathcal{L}_1 = \mathcal{L} \otimes \mathcal{O}_X(-P_0 - Q_0 - P_1) = \mathcal{O}_X(2P_0 - Q_0 - P_1), \\ \mathcal{L}_2 = \mathcal{L} \otimes \mathcal{O}_X(-2Q_0) = \mathcal{O}_X(3P_0 - 2Q_0). \end{cases}$$

Set

$$\begin{cases} s_0 = \dfrac{1}{\sqrt{3\alpha}}\dfrac{\zeta-\alpha}{\zeta}, & s_1 = \dfrac{1}{\sqrt{3\alpha}}\dfrac{\zeta-\alpha}{\zeta^2}, \\ s_2 = \sqrt{\dfrac{1-\alpha^2}{3\alpha}}\dfrac{1}{\zeta^2}, & s_3 = \sqrt{\dfrac{1}{3\alpha}}\dfrac{(1-\alpha\zeta)}{\zeta^3}. \end{cases}$$

We see that s_0 is a global section of $\mathcal{L}_0$, s_1 is a global section of $\mathcal{L}_1$ and s_2, s_3 global sections of $\mathcal{L}_2$. Evaluating $h(s_i, s_j)$ at $\lambda = 0(\pi^{-1}(0) = \{0,0,0,\alpha\})$, we easily see that $s_0, \cdots, s_3$ are orthonormal basis of $H^0(X, \mathcal{L})$. A $s_i(z)$ is given by $s_i(z) = e^{a(z,\zeta)}s_i$ for $i = 0, \cdots, 3$, where $a(z,\zeta) = \zeta^{-1}z - \zeta\overline{z}$. We obtain $F(z,1) = \mathrm{Ad}M(0)^{-1}\mathrm{diag}(a(z,\zeta_0), \cdots, a(z,\zeta_3))$, where $\{\zeta_0, \cdots, \zeta_3\} = \pi^{-1}(1)$.

Example 2.3. We will give an example of which the spectral curve is an elliptic curve. First of all, we will address some fundamental facts on elliptic functions.
[*Weierstrass zeta-function*]
Let $\mathbf{L} = \mathbf{Z} \oplus \tau\mathbf{Z}$, where τ is a complex number with $\mathrm{Im}(\tau) > 0$. The *Weierstrass zeta-function* $\zeta_w(u)$ is defined by

$$\zeta_w(u) = \frac{1}{u} + \sum_{\omega \in \mathbf{L}\setminus(0,0)} \{\frac{1}{(u-\omega)} + \frac{u}{\omega^2} + \frac{1}{\omega}\},$$

which has a pole of order 1 at $u = 0$. Set

$$\mathcal{P}(u) = -\frac{d}{du}\zeta_w(u) \quad .$$

This function uniformly converges on each compact subset and it is called *Weierstrass $\mathcal{P}-$function.* We have

$$\frac{d}{du}\mathcal{P}(u) = -2\sum_{\omega \in \mathbf{L}} \frac{1}{(u-\omega)^3}.$$

The definition of the summation means that $\frac{d}{du}\mathcal{P}(u)$ is invariant under the translations $u \to u+1$ and $u \to u+\tau$. Hence, $\frac{d}{du}\mathcal{P}(u)$ is doubly-periodic function. Therefore we may set

$$\begin{cases} \mathcal{P}(u+1) - \mathcal{P}(u) = c_1 \\ \mathcal{P}(u+\tau) - \mathcal{P}(u) = c_2 \end{cases}$$

where c_1, c_2 are some complex numbers. On the other hand, since $\mathcal{P}(u)$ is obviously even function, by setting $u = -1/2$ or $u = -\tau/2$ in the above equations we have $c_1 = c_2 = 0$. Therefore we see that the Weierstrass $\mathcal{P}-$function is doubly-periodic with periods $1, \tau$. Integrating $\mathcal{P}-$function, we have

$$\begin{cases} \zeta_w(u+1) - \zeta_w(u) = A \\ \zeta_w(u+\tau) - \zeta_w(u) = B \end{cases}$$

where A, B are some complex numbers. Notice that the residue theorem yields that

$$\frac{1}{2\pi\sqrt{-1}}\int_{\partial P_a}\zeta_w(u)du = 1$$

where P_a is some fundamental domain of the torus $\mathbf{R}^2/\mathbf{L}$. The integration of $\zeta_w(u)$ on P_a turns out to be $A\tau - B$, which yields, so-called, *Legendre's relation* :

$$A\tau - B = 2\pi\sqrt{-1}.$$

[*Jacobi's 1st theta function*]
Let $p(u) = \exp(\pi\sqrt{-1}u), q = \exp(\pi\sqrt{-1}\tau)$. Then the *Jacobi's 1st theta function* $\theta_1(u)$ is defined by

$$\theta_1(u) = \sqrt{-1}\sum_{n\in\mathbf{Z}}(-1)^n p(u)^{2n-1}q^{(n-\frac{1}{2})^2}.$$

By changing $n \to -n+1$ in the summation we see that $\theta_1(u)$ is an odd function. In particular, we have $\theta_1(0) = 0$. Moreover, since $(n-\frac{1}{2})^2+(2n-1) = (n+\frac{1}{2})^2-1$, the definition of the summation gives the following relations :

$$\text{(2.1)} \qquad \begin{cases} \theta_1(u+1) = -\theta_1(u) \\ \theta_1(u+\tau) = -p(u)^{-2}q^{-1}\theta_1(u) \end{cases}$$

With these facts in mind, we may construct a meromorphic function on some elliptic curve.

Let $X = \mathbf{R}^2/\mathbf{L}$ be a two-torus with lattice $\mathbf{L} = \mathbf{Z} \oplus \sqrt{-1}t\mathbf{Z}$, where t is some positive real number. In this case, we have $\overline{\zeta_w(u)} = \zeta_w(\overline{u})$. Define a function $\psi(z, \overline{z}, u)$ on X by

(2.2)

$$\psi(z,\overline{z},u) = \exp((\zeta_w(u-P_0) - Au)z - (\zeta_w(u-Q_0) - Au)\overline{z})\times$$
$$\frac{\theta_1(u-F_1)\cdots\theta_1(u-F_k)\theta_1(u-P_0)^m\theta_1(u-P_1)\cdots\theta_1(u-P_{n-m})\theta_1(u-G)}{\theta_1(u-E_1)\theta_1(u-E_2)\cdots\theta_1(u-E_{n+k+1})}$$

where

$$\begin{cases} \mathcal{L} \cong \mathcal{O}_X(D), \qquad D = \sum_{i=1}^{n+k+1} E_i - \sum_{i=1}^{k} F_i, \\ G = D - mP_0 - \sum_{i=1}^{n-m} P_i + z - \overline{z}\ . \end{cases}$$

Hence, $\mathcal{L}$ is a divisor line bundle of degree $(n+1)$. It follows from (2.1) that $\psi(z,\overline{z},u+1)=\psi(z,\overline{z},u), \psi(z,\overline{z},u+\sqrt{-1}t)=\psi(z,\overline{z},u)$, i.e., $\psi(z,\overline{z},u)$ is a meromorphic function on X with fixed $z,\overline{z}$. Moreover, since ψ behaves like $\exp(z\zeta^{-1}+O(\zeta))$ near P_0 and behaves like $\exp(-\overline{z}\zeta+O(1/\zeta))$ near Q_0 and ψ has a divisor $-D$ on $X\setminus\{P_0,Q_0\}$,we see that $\psi(z,\overline{z},u)\theta_A$ belongs to $H^0(X,\mathcal{O}_X(D)\otimes L(a))$(see [T]).

Now, consider a function

$$g(u)=\exp(4\pi\sqrt{-1}u)\Big(\frac{\theta_1(u-R_1)}{\theta_1(u-R_2)}\Big)^4$$

on X, where $R_1=\sqrt{-1}/4, R_2=3\sqrt{-1}/4=\rho_X(R_1)$ and $\rho_X(P)=\overline{P} \mod \mathbf{L}$. It follows from (2.1) that $g(u)$ is a meromorphic function on X. Define a covering map $\pi : X \longrightarrow \mathbf{C}P^1$ by $\pi(u)=g(u)/g(\sqrt{-1}/2)$. Then we have

$$\begin{cases}(\pi)=4(R_1)-4(R_2)\\ \ R=3(R_1)+(R_3)+(\rho_X(R_3))+3(R_2)\end{cases}$$

for some point $R_3\in X\setminus X_{\mathbf{R}}$. Therefore, we have $P_0=R_1, Q_0=R_2$. Let $\mathcal{L}=\mathcal{O}_X(3R_1+R_3)$ be a line bundle over X of degree 4. We see that $\pi^{-1}(1)=\{0,1/2,\sqrt{-1}/2,1/2+\sqrt{-1}/2\}$. It follows that each point of $\pi^{-1}(1)$ is fixed by the real structure ρ_X. In this case, we have $\mathcal{L}_0=\mathcal{O}_X(R_3), \mathcal{L}_1=\mathcal{O}_X(R_1+R_3-R_2), \mathcal{L}_2=\mathcal{O}_X(2R_1+R_3-2R_2), \mathcal{L}_3=\mathcal{O}_X(3R_1+R_3-3R_2)$. Set $\eta_0=0, \eta_1=1/2, \eta_2=\sqrt{-1}/2, \eta_3=1/2+\sqrt{-1}/2$. By choosing some constants c_0,c_1,c_2,c_3, we may define an orthonormal basis $\{s_i\}$ of $H^0(X,\mathcal{L})$ by

$$s_i=c_i\frac{\theta_1(u-\eta_0)\cdots\theta_1(u-\hat{\eta_i})\cdots\theta_1(u-\eta_3)}{\theta_1(u-R_1)^3\theta_1(u-R_3)},$$

where $\hat{\eta_i}=3R_1+R_3-(\eta_0+\cdots\eta_{i-1}+\eta_{i+1}+\cdots+\eta_3)$. In fact, it follows from the positive definiteness of h that $\hat{\eta_i}\neq\eta_i$. Now, we set

$$\begin{cases} s_0(z)=\dfrac{1}{c_0}\dfrac{\theta_1(u-R_1)^3\theta_1(u-R_3-z-\overline{z})}{\theta_1(u-\hat{\eta_0})\theta_1(u-\eta_1)\theta_1(u-\eta_2)\theta_1(u-\eta_3)}\exp(f(z,u)),\\ s_1(z)=\dfrac{1}{c_1}\dfrac{\theta_1(u-R_1)^2\theta_1(u-R_2)\theta_1(u-R_1-R_3+R_2-z-\overline{z})}{\theta_1(u-\eta_0)\theta_1(u-\hat{\eta_1})\theta_1(u-\eta_2)\theta_1(u-\eta_3)}\exp(f(z,u)),\\ s_2(z)=\dfrac{1}{c_2}\dfrac{\theta_1(u-R_1)\theta_1(u-R_2)^2\theta_1(u-2R_1-R_3+2R_2-z-\overline{z})}{\theta_1(u-\eta_0)\theta_1(u-\eta_1)\theta_1(u-\hat{\eta_2})\theta_1(u-\eta_3)}\exp(f(z,u)),\\ s_3(z)=\dfrac{1}{c_3}\dfrac{\theta_1(u-R_2)^3\theta_1(u-3R_1-R_3+3R_2-z-\overline{z})}{\theta_1(u-\eta_0)\theta_1(u-\eta_1)\theta_1(u-\eta_2)\theta_1(u-\hat{\eta_3})}\exp(f(z,u)),\end{cases}$$

where $f(z,u)=\exp((\zeta_w(u-R_1)-Au)z-(\zeta_w(u-R_2)-Au)\overline{z})$. Then $F(z,1)=M^{-1}\cdot M(z)$ is computable and the first column vector of $F(z,1)$ gives a harmonic map of $\mathbf{R}^2$ into $\mathbf{C}P^3$ with isotropy order 3(i.e., superconformal harmonic map).

3. Non-isotropic harmonic maps of $\mathbf{R}^2$ into $G_2(\mathbf{C}^4)$

Represent $G_2(\mathbf{C}^4)$ as $G_2(\mathbf{C}^4) = G/K$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. We have a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. Set $\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}$. The equation $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ is the integrability condition for $\alpha_\lambda = F(z,\lambda)^{-1}dF(z,\lambda)$ with $F(z,1) : \mathbf{R}^2 \longrightarrow U(4)$. If $\alpha_\lambda(\partial/\partial z)$ is a holomorphic matrix for fixed complex coordinate z then $[\alpha'_\lambda, \alpha''_\lambda] = 0$, where α'_λ(resp. α''_λ) is the (1,0)-part (resp. (0,1)-part) of the 1-form α_λ. Therefore, α'_λ is a normal matrix. We construct examples of such properties from spectral data.

Let $\pi : X \longrightarrow \mathbf{P}^1_\lambda$ be defined by $\pi(\zeta) = \zeta^4 = \lambda$. Define $\mathcal{L}$ by $\mathcal{L} = \mathcal{O}_X(3(0))$. For $\tau, \mu, \nu, \kappa \in \mathbf{C}$, define $a(z,\zeta) \in \mathfrak{g}_{\mathbf{R}}$ by

$$(3.1) \qquad \begin{cases} a(z,\zeta) = b(z,\zeta) - \overline{\rho_{X*}(b(z,\zeta))} \\ b(z,\zeta) = (\tau + \mu\zeta^{-1} + \nu\zeta^{-2} + \kappa\zeta^{-3})z \end{cases}.$$

Set $s_i(z) = \exp(a(z,\zeta))\zeta^{-i}$ and $s_i = s_i(0)$ for $i = 0, \cdots, 3$. Define $F(z,\lambda)$ by

$$(s_0(z), \cdots, s_3(z)) = (s_0, \cdots, s_3)F(z,\lambda).$$

We have

$$F(z,\lambda)^{-1}\partial F(z,\lambda) = \begin{pmatrix} \tau & \kappa\lambda^{-1} & \nu\lambda^{-1} & \mu\lambda^{-1} \\ \mu & \tau & \kappa\lambda^{-1} & \nu\lambda^{-1} \\ \nu & \mu & \tau & \kappa\lambda^{-1} \\ \kappa & \nu & \mu & \tau \end{pmatrix}.$$

To investigate whether $F(z,1)$ is a framing of a harmonic map or not, we set

$$A_\lambda = \begin{pmatrix} \tau & \kappa & \nu\lambda^{-1} & \mu\lambda^{-1} \\ \mu & \tau & \kappa\lambda^{-1} & \nu\lambda^{-1} \\ \nu\lambda^{-1} & \mu\lambda^{-1} & \tau & \kappa \\ \kappa\lambda^{-1} & \nu\lambda^{-1} & \mu & \tau \end{pmatrix}.$$

Set $\alpha_\lambda = A_\lambda - (A_{\overline{\lambda}^{-1}})^*$.

Lemma 3.1. *$\alpha_\lambda(\partial/\partial z)$ is a normal matrix for each $\lambda \in \mathbf{C}^*$ if and only if $\kappa\overline{\mu} = \overline{\kappa}\mu$.*

Therefore, we assume that $\kappa\overline{\mu} = \overline{\kappa}\mu$. Let $\zeta_0, \cdots, \zeta_3$ be $1, -1, i, -i$, respectively, which are elements of $\pi^{-1}(1)$. If we set $M(z) = \begin{pmatrix} s_0(z)\mid_{\zeta_0} & \cdots & s_3(z)\mid_{\zeta_0} \\ \vdots & \ddots & \vdots \\ s_0(z)\mid_{\zeta_3} & \cdots & s_3(z)\mid_{\zeta_3} \end{pmatrix}$ then we have $M(z) = \mathrm{diag}(\exp(a(z,\zeta_0)), \cdots, \exp(a(z,\zeta_3))) \cdot M(0)$ and the framing of the corresponding harmonic map is given by

$$\begin{cases} F(z,1) = \mathrm{Ad}M(0)^{-1}\mathrm{diag}(\exp(a(z,\zeta_0)), \cdots, \exp(a(z,\zeta_3))), \\ M(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix}. \end{cases} \tag{3.2}$$

The corresponding harmonic map $\mathbf{R}^2 \longrightarrow G_2(\mathbf{C}^4)$ is $\mathbf{R}^2$-equivariant and non-isotropic.

Proposition 3.1. *$F(z,1)$ given by (3.2) is a framing of a non-isotropic harmonic map $\varphi : \mathbf{R}^2 \longrightarrow G_2(\mathbf{C}^4)$. Moreover, the following hold :*
(1) φ is weakly conformal if and only if $\nu^2 + \mu\kappa \equiv 0$,
(2) φ is irreducible and of non-semisimple type if and only if either $\mu \equiv 0, \nu\kappa \neq 0$ or $\kappa \equiv 0, \mu\nu \neq 0$.

4. Non-isotropic harmonic map of non-semisimple type

Suppose that $\kappa \equiv 0$ in the previous section. In this case, ∂'-first return map A^{FR} of the harmonic sequence of the harmonic map $p \circ F(z,1)$ is represented by non-semisimple, rank 2-matrix, where $p : U(4) \longrightarrow G_2(\mathbf{C}^4)$ is the coset projection(see [U1]). In the following, we consider the case where A^{FR} is a non-semisimple rank 2-matrix. We assume that $\alpha(\partial/\partial z)$ is a constant matrix for some fixed complex coordinate z. Then, the harmonic map equation is written as

$$\begin{cases} [\alpha'_{\mathfrak{k}}(\partial/\partial z), \alpha''_{\mathfrak{m}}(\partial/\partial \overline{z})] = 0, \\ [\alpha'_{\mathfrak{k}}(\partial/\partial z), \alpha''_{\mathfrak{k}}(\partial/\partial \overline{z})] + [\alpha'_{\mathfrak{m}}(\partial/\partial z), \alpha''_{\mathfrak{m}}(\partial/\partial \overline{z})] = 0. \end{cases} \tag{4.1}$$

Clearly, the equations in (4.1) are invariant under $\mathrm{Ad}(U(2) \times U(2))$-action on $\alpha'(\partial/\partial z)$. Choose unitary basis $\mathbf{e}_0, \cdots, \mathbf{e}_3$ of $\mathbf{C}^4$ such that $\alpha'(\partial/\partial z)$ is represented by

$$\alpha'(\partial/\partial z) = \begin{pmatrix} a_0 & 0 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ 0 & d_1 & d_2 & d_3 \end{pmatrix}.$$

The first equation in (4.1) yields

$$\begin{cases} c_3 = 0, a_0 = c_2, b_0\overline{b}_2 = 0, d_3 = b_1, b_1\overline{b}_2 = c_2\overline{b}_2, \\ \overline{b}_2 d_2 = 0, c_2\overline{c}_1 + d_2\overline{d}_1 = b_0\overline{c}_0 + b_1\overline{c}_1, a_0\overline{a}_3 + b_0\overline{b}_3 = d_2\overline{a}_2 + d_3\overline{a}_3. \end{cases} \tag{4.2}$$

If $b_2 \neq 0$, then $b_0 = d_2 = 0, b_1 = c_2$. Therefore, we see that $\alpha'_{\mathfrak{k}}(\partial/\partial z)$ is a scalar matrix, which, together with second equation in (4.1), means that $\alpha'_{\mathfrak{m}}(\partial/\partial z)$ is semisimple. Thus, A^{FR} is semisimple. Hence, we may assume that $b_2 = 0$. This, together with (4.2) and the second equation in (4.1), yields

(4.3)
$$\begin{cases} \qquad\qquad a_1 = b_2 = c_3 = d_0 = 0, \\ \quad \overline{c}_1(a_0 - b_1) = b_0\overline{c}_0 - d_2\overline{d}_1, \quad \overline{a}_3(a_0 - b_1) = \overline{a}_2 d_2 - b_0\overline{b}_3, \\ \quad \overline{b}_0(a_0 - b_1) = \overline{c}_0 c_1 - a_3\overline{b}_3, \quad \overline{d}_2(a_0 - b_1) = \overline{a}_2 a_3 - c_1\overline{d}_1, \\ \mid b_0 \mid^2 + \mid c_0 \mid^2 = \mid a_2 \mid^2 + \mid a_3 \mid^2, \quad \mid c_1 \mid^2 + \mid d_1 \mid^2 = \mid b_0 \mid^2 + \mid b_3 \mid^2, \\ \mid a_2 \mid^2 + \mid d_2 \mid^2 = \mid c_0 \mid^2 + \mid c_1 \mid^2 . \end{cases}$$

The first return map is given by $A^{FR} = \begin{pmatrix} a_2c_0 & a_2c_1 + a_3d_1 \\ 0 & b_3d_1 \end{pmatrix}$. Hence, A^{FR} is non-semisimple, rank 2-matrix if and only if

$$a_2c_0 = b_3d_1, \quad a_2c_1 + a_3d_1 \neq 0. \tag{4.4}$$

By the equations in the second line in (4.3) and the first equation in (4.4), we have $(\overline{b}_3\overline{c}_1 + \overline{c}_0\overline{a}_3)(a_0 - b_1) = 0$, hence $a_0 = b_1$ or $a_3c_0 + b_3c_1 \equiv 0$. Suppose that $a_0 \neq b_1$. Then the first equation in (4.4) and $a_3c_0 + b_3c_1 = 0$ yield that

$$a_2c_1 + a_3d_1 = c_1\frac{b_3d_1}{c_0} + d_1(-\frac{b_3c_1}{c_0}) = 0,$$

which contradicts the second equation in (4.4). Therefore, we must have $a_0 = b_1$. In this case, by (4.3) we can set

$$r := \frac{d_1}{c_0} = \frac{a_2}{b_3} = \frac{\overline{b_0}}{\overline{d_2}}, \quad s := \frac{c_1}{a_3} = \frac{\overline{b_3}}{\overline{c_0}}. \tag{4.5}$$

Note that any denominator of the equations in (4.5) is non-zero by the assumptions on A^{FR}. By the equations in the fourth line of (4.3) we see that $\mid s \mid= 1$ or $\mid a_3 \mid^2 = \mid c_0 \mid^2 + \mid d_1 \mid^2$. Each case is considered separately.

Case 1 (case of $\mid s \mid= 1$)

In this case, the equations in the fourth and fifth lines of (4.3) mean that $\mid r \mid= 1$ or $\mid d_2 \mid^2 = 2 \mid c_0 \mid^2$. First, consider the case where $\mid r \mid= 1$. Set $s = \exp(i\theta), r = \exp(i\varphi)$. The equation in the fourth line of (4.3) yields that $\mid d_2 \mid^2 = \mid a_3 \mid^2$ and hence we may set $d_2 = a_3\exp(i\hat{\theta})$, where $\hat{\theta}$ is a real number. Setting $\tau = a_0, \mu = d_2\exp(-i\varphi)$, $\nu = c_0\exp(-i(\varphi+\theta-\hat{\theta}))$, $\lambda^{-1} = \exp(\frac{i}{2}(3\varphi+\theta-2\hat{\theta}))$ and choosing new unitary basis $\hat{\mathbf{e}}_0 = \exp(i(\varphi-\theta)/2)\mathbf{e}_0, \hat{\mathbf{e}}_1 = \exp(i(\varphi-\theta)/2)\mathbf{e}_1, \hat{\mathbf{e}}_2 = \mathbf{e}_2, \hat{\mathbf{e}}_3 = \exp(i\varphi)\mathbf{e}_3$, we see that

(4.6)
$$\alpha'(\partial/\partial z) = \lambda^{-1}\begin{pmatrix} 0 & 0 & nu & \mu \\ 0 & 0 & 0 & \nu \\ \nu & \mu & 0 & 0 \\ 0 & \nu & 0 & 0 \end{pmatrix} + \begin{pmatrix} \tau & 0 & 0 & 0 \\ \mu & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & \mu & \tau \end{pmatrix}.$$

Next, consider the case where $\mid d_2 \mid^2 = 2 \mid c_0 \mid^2$. We have $\mid b_0 \mid^2 = \mid d_2 \mid^2 \mid r \mid^2 = 2 \mid c_0 \mid^2 \mid r \mid^2 = 2 \mid d_1 \mid^2$. Moreover, $\mid s \mid = 1$ means that $\mid c_1 \mid^2 = \mid a_3 \mid^2$ and $\mid b_3 \mid^2 = \mid c_0 \mid^2$. Therefore, the equation $\mid c_1 \mid^2 + \mid d_1 \mid^2 = \mid b_0 \mid^2 + \mid b_3 \mid^2$ in (4.3) gives an equation $\mid a_3 \mid^2 = \mid c_0 \mid^2 + \mid d_1 \mid^2$. Thus, this case is a special case of Case 2 below.

Case 2. (case of $\mid a_3 \mid^2 = \mid c_0 \mid^2 + \mid d_1 \mid^2$)

In this case, by the equations in the fourth line of (4.3) we obtain

$$\begin{cases} \mid a_3 \mid^2 = \mid c_0 \mid^2 (\mid r \mid^2 + 1), \\ \mid d_2 \mid^2 = \mid c_0 \mid^2 (\mid s \mid^2 + 1), \end{cases}$$

which, together with $a_3 = c_1 s^{-1}, d_2 = b_0 \overline{r}^{-1}$, yield

(4.7)
$$\begin{cases} \mid c_1 \mid^2 = \mid c_0 \mid^2 \mid s \mid^2 (\mid r \mid^2 + 1), \\ \mid b_0 \mid^2 = \mid c_0 \mid^2 \mid r \mid^2 (\mid s \mid^2 + 1). \end{cases}$$

By (4.7), we can set

$$\begin{aligned} r &= \hat{r}\exp(i\varphi), \quad s = \hat{s}\exp(i\theta), \\ c_1 &= \hat{s}\sqrt{\hat{r}^2+1}c_0\exp(i\psi), \quad b_0 = \hat{r}\sqrt{\hat{s}^2+1}c_0\exp(i\gamma), \end{aligned}$$

where $\hat{r} = \mid r \mid, \hat{s} = \mid s \mid$ and $\varphi, \theta, \psi, \gamma \in \mathbf{R}$. Then, choosing new basis $\hat{\mathbf{e}}_0 = \exp(\frac{i}{2}(\varphi-\theta))\mathbf{e}_0, \hat{\mathbf{e}}_1 = \exp(\frac{i}{2}(\varphi-\theta-2\psi))\mathbf{e}_1, \hat{\mathbf{e}}_2 = \mathbf{e}_2, \hat{\mathbf{e}}_3 = \exp(i(\varphi-\psi))\mathbf{e}_3$ and setting $\tau = a_0, \delta = c_0\exp(i(\gamma+\psi)), \lambda^{-1} = \exp(\frac{i}{2}(\varphi-\theta-2\gamma-2\psi))$ we see that

(4.8)
$$\begin{aligned} \alpha'(\partial/\partial z) =& \lambda^{-1}\begin{pmatrix} 0 & 0 & \hat{r}\hat{s}\delta & \sqrt{\hat{r}^2+1}\delta \\ 0 & 0 & 0 & \hat{s}\delta \\ \delta & \hat{s}\sqrt{\hat{r}^2+1}\delta & 0 & 0 \\ 0 & \hat{r}\delta & 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} \tau & 0 & 0 & 0 \\ \hat{r}\sqrt{\hat{s}^2+1}\delta & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & \sqrt{\hat{s}^2+1}\delta & \tau \end{pmatrix}. \end{aligned}$$

Thus, we obtain

Proposition 4.1. *Let α be the pull-back of the Maurer-Cartan form by $F : \mathbf{R}^2 \longrightarrow U(4)$. Assume that the first return map A^{FR} of the harmonic map $p \circ F : \mathbf{R}^2 \longrightarrow G_2(\mathbf{C}^4)$ is represented by non-semisimple, rank 2-matrix and that $\alpha'(\partial/\partial z)$ is a constant matrix with respect to some fixed complex coordinate system. Then $\alpha'(\partial/\partial z)$ is of the form (4.6) or (4.8) under the* $\mathrm{Ad}(U(2) \times U(2))$*-action.*

5. Dressing action on harmonic map and spectral data

First of all, we review the definition of the dressing actions (see [BP2]). Let G be a compact semisimple Lie group with an automorphism τ of finite order $k(\geq 2)$, of which fixed set K. We fix an Iwasawa decomposition of the reductive group $K^{\mathbf{C}} : K^{\mathbf{C}} = KB$, where B is a solvable subgroup of $K^{\mathbf{C}}$. Fix $0 < \epsilon < 1$. For a Riemann sphere $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$, define open subsets by

$$I_\epsilon = \{\lambda \in \mathbf{P}^1 :\mid \lambda \mid < \epsilon\}, \qquad I_{1/\epsilon} = \{\lambda \in \mathbf{P}^1 :\mid \lambda \mid > 1/\epsilon\},$$
$$E^{(\epsilon)} = \{\lambda \in \mathbf{P}^1 : \epsilon < \mid \lambda \mid < 1/\epsilon\}.$$

Let C_ϵ and $C_{1/\epsilon}$ be the circles of radius ϵ and $1/\epsilon$ about $0 \in \mathbf{C}$, respectively. Setting $I^{(\epsilon)} = I_\epsilon \cup I_{1/\epsilon}$ and $C^{(\epsilon)} = C_\epsilon \cup C_{1/\epsilon}$ we have $\mathbf{P}^1 = I^{(\epsilon)} \cup C^{(\epsilon)} \cup E^{(\epsilon)}$. Define the group of smooth maps $\Lambda^\epsilon G_\tau$ by

$$\Lambda^\epsilon G_\tau = \{g : C^{(\epsilon)} \to G^{\mathbf{C}} \mid g(\omega\lambda) = \tau g(\lambda), \overline{g(\lambda)} = g(1/\overline{\lambda}) \text{ for all } \lambda \in C^{(\epsilon)}\}.$$

Define some subgroups of $\Lambda^\epsilon G_\tau$ by

$$\Lambda^\epsilon_E G_\tau = \{g \in \Lambda^\epsilon G_\tau \mid g \ \text{ extends holomorphically to } \ g : E^{(\epsilon)} \to G^{\mathbf{C}}\},$$
$$\Lambda^\epsilon_I G_\tau = \{g \in \Lambda^\epsilon G_\tau \mid g \ \text{ extends holomorphically to } \ g : I^{(\epsilon)} \to G^{\mathbf{C}}\}.$$

Any element g of these subgroups satisfies $\overline{g(\lambda)} = g(1/\overline{\lambda}), g(\omega\lambda) = \tau g(\lambda)$ for all λ in its domain of definition. If $g \in \Lambda^\epsilon_I G_\tau$ then $g(0) \in K^{\mathbf{C}}$. Hence, we define the subgroup

$$\Lambda^\epsilon_{I,B} G_\tau = \{g \in \Lambda^\epsilon_I G_\tau \mid g(0) \in B\}.$$

Then, McIntosh[M1] proved that multiplication $\Lambda^\epsilon_E G_\tau \times \Lambda^\epsilon_{I,B} G_\tau \longrightarrow \Lambda^\epsilon G_\tau$ is a diffeomorphism onto. This is, what we call, Iwasawa decomposition of $\Lambda^\epsilon G_\tau$. Any $g \in \Lambda^\epsilon G_\tau$ has a unique factorization $g = g_E g_I$, where $g_E \in \Lambda^\epsilon_E G_\tau, g_I \in \Lambda^\epsilon_{I,B} G_\tau$. The action of $\Lambda^\epsilon_I G_\tau$ on $\Lambda^\epsilon_E G_\tau$ is given by

$$g \sharp_\epsilon h = (gh)_E,$$

for $g \in \Lambda_I^\epsilon G_\tau, h \in \Lambda_E^\epsilon G_\tau$. We call this action *dressing action* on $\Lambda_E^\epsilon G_\tau$.

[**Dressing action on extended framing**]

Define $\Lambda_{\mathrm{hol}} G_\tau$ by

$$\Lambda_{\mathrm{hol}} G_\tau = \{g : \mathbf{C}^* \to G^{\mathbf{C}} \mid g \text{ is holomorphic and } \overline{g(\lambda)} = g(1/\overline{\lambda}), g(\omega\lambda) = \tau g(\lambda)\}.$$

Then we have

$$\Lambda_{\mathrm{hol}} G_\tau = \bigcap_{0<\epsilon<1} \Lambda_E^\epsilon G_\tau.$$

A map $F : \mathbf{R}^2 \longrightarrow \Lambda_{\mathrm{hol}} G_\tau$ is an *extended framing* if $F^{-1}dF = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}$. For an extended framing $F(z,\lambda)$ and $g(\lambda) \in \Lambda_I^\epsilon G_\tau$, define $(g\sharp F)(z,\lambda) : \mathbf{R}^2 \longrightarrow \Lambda_E^\epsilon G_\tau$ by

$$(g\sharp F)(z,\lambda) = g(\lambda)\sharp(F(z,\lambda)).$$

Then $g\sharp F$ is also an extended framing. We may write

$$g(\lambda)F(z,\lambda) = \hat{F}(z,\lambda)B(z,\lambda) \tag{5.1}$$

for some $B(z,\lambda) : \mathbf{R}^2 \longrightarrow \Lambda_{I,B}^\epsilon G_\tau$ and extended framing $\hat{F}(z,\lambda)$.

[**Dressing action on spectral data**]

Recall that we may represent $(s_0(z), \cdots s_n(z)) = (s_0, \cdots, s_n)F(z,\lambda)$. We may also represent $(\hat{s}_0(z), \cdots, \hat{s}_n(z)) = (\hat{s}_0, \cdots, \hat{s}_n)\hat{F}(z,\lambda)$, where $\hat{F} = g\sharp F$ for some $g \in \Lambda_I^\epsilon G_\tau$. Now, we use (5.1). We suppose that $F(0,\lambda) = \hat{F}(0,\lambda) = I$. Notice that $B(0,\lambda) = g(\lambda)$ in this case. We compute

$$\begin{aligned}(s_0(z), \cdots, s_n(z))B(z,\lambda)^{-1} &= (s_0(z), \cdots, s_n(z))F(z,\lambda)^{-1}g(\lambda)^{-1}\hat{F}(z,\lambda) \\ &= (s_0, \cdots, s_n)B(0,\lambda)^{-1}\hat{F}(z,\lambda).\end{aligned}$$

Hence, we may take $(\hat{s}_0(z), \cdots, \hat{s}_n(z)) = (s_0(z), \cdots, s_n(z))B(z,\lambda)^{-1}$. This is a dressing action on spectral data. By the results in [BP2] and [M2], we have

Theorem 5.1. *A spectral data for non-isotropic harmonic map of finite type with isotropy order r :* $\mathbf{R}^2 \longrightarrow \mathbf{C}P^n$ *is obtained by dressing action on the spectral data of genus zero with isotropy order r, of which the spectral curve is given by* $\lambda = \zeta^{r+1}\frac{(\zeta-\alpha_1)\cdots(\zeta-\alpha_{n-r})}{(\zeta-\beta_1)\cdots(\zeta-\beta_{n-r})}$, *where* $\beta_j = \rho_X(\alpha_j)$ *and* $0 <\mid \alpha_j \mid< 1$ *for* $j = 1, \cdots, n-r$.

Proof. Since dressing action on extended framing preserves the isotropy order of the corresponding non-isotropic harmonic map, the result follows from the fact that non-isotropic harmonic map of finite type $\mathbf{R}^2 \longrightarrow \mathbf{C}P^n$ is obtained by dressing action on the vacuum solution(see [BP2]). The vacuum solution is $\mathbf{R}^2$-equivariant and its spectral curve has genus zero(see [M2]) and Theorem 8 in [M2] and a result in [T] imply that we may take the spectral curve of genus zero of the form in our theorem. q.e.d.

Theorem 5.2. *Let $\varphi : \mathbf{R}^2 \longrightarrow G_2(\mathbf{C}^4)$ be a harmonic map of non-semisimple type. Let $F : \mathbf{R}^2 \longrightarrow U(4)$ be a framing of φ. Assume that $F^{-1}\partial F(\partial/\partial z)$ is constant with respect to some fixed complex coordinate system. Then F is congruent, up to* $\mathrm{Ad}(U(2)\times U(2))$*-action, to (3.2) with $\kappa \equiv 0$ or to a dressing orbit of (3.2) with $\kappa \equiv 0$ and $\nu = c\mu$ for some positive constant c.*

Proof. We only have to show that (4.8) is obtained by dressing action of (4.6). Write

$$g(\lambda) = g_0 + g_1\lambda + g_2\lambda^2 + \cdots ,$$
$$B(z,\lambda) = B_0(z) + B_1(z)\lambda + B_2(z)\lambda^2 + \cdots .$$

Hence, we have $g_j = B_j(0)$ for $j = 0, 1, \cdots$. Let $\alpha'_{\mathfrak{m}}$ and $\alpha'_{\mathfrak{k}}$ be λ^{-1}-coefficient matrix and λ^0-coefficient matrix of $\alpha'(\partial/\partial z)$ in (4.6). In the same way, let $\hat{\alpha}'_{\mathfrak{m}}$ and $\hat{\alpha}'_{\mathfrak{k}}$ be λ^{-1}- coefficient matrix and λ^0-coefficient matrix of $\alpha'(\partial/\partial z)$ in (4.8). We know that $\alpha''_{\mathfrak{m}} = -(\alpha'_{\mathfrak{m}})^*, \alpha''_{\mathfrak{k}} = -(\alpha'_{\mathfrak{k}})^*$ hold. The same relations hold for $\hat{\alpha}''_{\mathfrak{m}}$ and $\hat{\alpha}''_{\mathfrak{k}}$. The relation (5.1) yields that

$$\begin{cases} \hat{\alpha}'_{\mathfrak{m}} = B_0\alpha'_{\mathfrak{m}}B_0^{-1}, \\ \hat{\alpha}''_{\mathfrak{k}} = B_0\alpha''_{\mathfrak{k}}B_0^{-1} - \overline{\partial}B_0 \cdot B_0^{-1}. \end{cases} \tag{5.2}$$

Suppose that $\nu = c\mu$ for some positive constant c. We choose δ and b by

$$\delta = \frac{\nu}{\sqrt{\hat{r}\hat{s}}}, \quad b = \Big(\frac{\sqrt{\hat{r}^2+1}(\hat{s}^2+1)c}{2\hat{r}\hat{s}\sqrt{\hat{s}}}\Big)^{1/2}.$$

Define $B_0(z)$ by

$$B_0(z) = \begin{pmatrix} \sqrt{\hat{r}\hat{s}}b & \sqrt{\hat{r}\hat{s}}f(z,\overline{z}) & 0 & 0 \\ 0 & \frac{1}{\sqrt{\hat{r}}b} & 0 & 0 \\ 0 & 0 & b & f(z,\overline{z}) + \frac{\sqrt{\hat{r}^2+1}(\hat{s}^2-1)}{2b\hat{r}\hat{s}\sqrt{\hat{s}}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{\hat{s}}b} \end{pmatrix},$$

where $f(z,\overline{z}) = (\frac{c\sqrt{\hat{s}^2+1}}{\sqrt{\hat{r}\hat{s}b}} - b)\mu z + (\frac{c\sqrt{\hat{s}^2+1}}{\sqrt{\hat{r}\hat{s}b}} - b)\overline{\mu z}$. Now, we can verify that the relations (5.2) hold for this chosen $B_0(z)$. This B_0 already determines $\hat{F}(z,\lambda)$. The uniqueness of Iwasawa decomposition and appropriate choice of $g_j, B_j (j = 1, 2, \cdots)$ yield the result. q.e.d.

6. Some generalizations of the previous construction

We define spectral data as follows :

Definition. (1) X : compact Riemann surface of genus p with real structure ρ_X,
(2) $\pi : X \longrightarrow \mathbf{P}^1_\lambda$ is a holomorphic covering map of degree $2n$ such that $\pi \circ \rho_X = \overline{\pi}^{-1}$ and

$$(\lambda \circ \pi) = 2(m+1)P_0 + E_0 - 2(m+1)Q_0 - E_\infty,$$

where
$E_0 = \sum_{i=1}^{2n-2m-2} P_i, E_\infty = \sum_{i=1}^{2n-2m-2} Q_i, Q_i = \rho_X(P_i)$ for $i = 0, 1, \cdots, 2n - 2m - 2$,
(3) $\mathcal{L}$ is a line bundle over X of degree $(2n-1+p)$ such that $f : \mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} \to \mathcal{O}_X(R)$ is ρ_X-equivariant isomorphism, where R is the ramification divisor for π,
(4) π has no branch points on S^1_λ, ρ_X fixes each point of $\pi^{-1}(S^1_\lambda)$ and f is non-negative on $\pi^{-1}(S^1_\lambda)$.
Define $\hat{\mathcal{L}}_0, \cdots, \hat{\mathcal{L}}_m$ by

$$\begin{cases} \hat{\mathcal{L}}_j = \mathcal{L} \otimes \mathcal{O}_X((2j-2m)P_0 - 2jQ_0 - E_0) & \text{for } j = 0, 1, \cdots, m-1, \\ \hat{\mathcal{L}}_m = \mathcal{L} \otimes \mathcal{O}_X(-2mQ_0). \end{cases}$$

The relation between these line bundles and line bundles defined in section 1 is given by

$$\begin{cases} \mathcal{L}_{2j} = \hat{\mathcal{L}}_j(-P_0), \quad \mathcal{L}_{2j+1} = \hat{\mathcal{L}}_j(-Q_0) & \text{for } j = 0, 1, \cdots, m-1, \\ \mathcal{L}_{2m} = \hat{\mathcal{L}}_m(-P_0 - E_0), \quad \mathcal{L}_{2m+1} = \hat{\mathcal{L}}_m(-Q_0). \end{cases}$$

Hence, the sheaves of germs of holomorphic sections of $\mathcal{L}_{2j}$ and $\mathcal{L}_{2j+1}$ are subsheaves of the sheaf of germs of holomorphic sections of $\hat{\mathcal{L}}_j$ for $j = 0, 1, \cdots, m$. For $\tau, \mu, \nu \in \mathbf{C}$, we define $a(z, \zeta)$ by

$$\begin{cases} a(z,\zeta) = b(z,\zeta) - \overline{\rho_{X*}b(z,\zeta)}, \\ b(z,\zeta) = (\tau + \mu\zeta^{-1} + \nu\zeta^{-2})z \quad . \end{cases}$$

In the same way as in section 3, we may construct non-isotropic, non-semisimple harmonic map $\mathbf{R}^2 \longrightarrow G_2(\mathbf{C}^{2n})$. Therefore, we obtain

Proposition 6.1. *There are many examples of non-isotropic, non-semisimple harmonic map of finite type* $\mathbf{R}^2 \longrightarrow G_2(\mathbf{C}^{2n})$.

Proof. It is only to prove that they are of finite type. However, it follows from the result in [OU]. q.e.d.

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Department of Mathematics, School of Medicine, Nihon University, Itabashi, Tokyo 173-0032, Japan

E-mail address: sudagawa@med.nihon-u.ac.jp