

ON A LOOP EXPANSION FORMULA FOR ENUMERATION OF EULER SUBGRAPHS OF A PLANE GRAPH

MAKOTO IDZUMI

(Received: January 17, 2000)

ABSTRACT. The Vdovichenko formula is a formula for counting Euler subgraphs of a labelled graph which is embedded into a plane. We interpret it as a loop expansion formula. The nature of the loop expansion is explored in detail.

1. INTRODUCTION

An Euler graph is a graph of which each vertex has even degree. If an Euler graph P is a subgraph of a graph G we say that P is an Euler subgraph of G .

Let G be a graph embedded into a plane, and consider the generating function for Euler subgraphs of the graph with a given number of edges (see (1) and (2)).

Vdovichenko [1] gave a nice formula for the square lattice graph G on a plane which expresses the generating function in terms of a “transition matrix” using an analogy with a random walk problem on graphs. In a recent paper the author [2] gave a generalization which is applicable for any graph G embedded into a plane. However, it seems that no rigorous proof of the original Vdovichenko formula (and so, the generalized one by the author) has been appeared in the literatures.

This paper presents a preparatory consideration on the formula(s) (hopefully) toward a rigorous proof. We shall clarify the geometric meaning of the Vdovichenko formula (in the generalized form, presented in [2]); the formula is interpreted as a loop expansion where loops are drawn on the graph; see (10). The nature of this loop expansion will be explored in Section 5.

2. TERMINOLOGY ON GRAPHS

To begin with we give precise definitions of Euler subgraphs, a generating function for them, and a plane graph.

1991 *Mathematics Subject Classification.* 05C30, 05C38, 05A15, 05A19, 82B20.

Key words and phrases. Combinatorial enumeration, Graphs, Loop expansion, Ising model.

2.1. A graph and its Euler subgraphs. Abstractly, a *graph* (or *digraph*) is a triple $G = (V(G), E(G), \phi)$ consisting of a finite set $V(G)$ of *vertices* and a finite set $E(G)$ of *edges*, and a mapping $\phi : E(G) \rightarrow V(G) \times V(G)$, $\phi(e) = (I(e), T(e))$. The vertices $I(e)$ and $T(e)$ are called the *initial vertex* and the *terminal vertex* of e , respectively, and they are called the *endpoints* of e . We allow closed edges (an edge joining a vertex to itself) and parallel edges (more than one edge joining two vertices, or one vertex in case of a closed edge).

Let G be a graph. By an *Euler subgraph* of G we shall mean a subgraph P which satisfies $V(P) = V(G)$, $E(P) \subset E(G)$, and

$$\deg_P(\alpha) \text{ is even for every } \alpha \in V(P)$$

where the *degree* of a vertex $\alpha \in V(P)$ with respect to P is defined by

$$\deg_P(\alpha) = \#\{e \in E(P) \mid I(e) = \alpha\} + \#\{e \in E(P) \mid T(e) = \alpha\}.$$

Note that Euler subgraphs are called *polygons* in the Ising model literatures.

Let $\mathcal{P}_p(G)$ denote the set of all Euler subgraphs with p edges and let $\mathcal{P}(G) = \bigcup_{p=0}^{\infty} \mathcal{P}_p(G)$ (this set is finite). Let $\mathbf{x} = \{x_e \mid e \in E(G)\}$ be a set of commutative variables (indeterminates). Consider a polynomial in these variables defined by

$$(1) \quad S(\mathbf{x}) = \sum_{P \in \mathcal{P}(G)} \prod_{e \in E(P)} x_e.$$

We shall call $S(\mathbf{x})$ the generating function for Euler subgraphs, for if we substitute $x_e = x$ for all $e \in E(G)$ we obtain a generating function for Euler subgraphs with a given number of edges:

$$(2) \quad [S(\mathbf{x})]_{\text{all } x_e = x} = \sum_{p=0}^{\infty} A_p x^p$$

where A_p is the number of Euler subgraphs with p edges.

2.2. A plane graph. Usually a graph G is regarded as a topological space: first we suppose that each vertex is homeomorphic to a point and each edge is homeomorphic to the segment $[0, 1]$ of the real line, and that the initial and the terminal vertices correspond to 0 and 1 in the segment respectively; consider a direct sum space $(\bigsqcup_{\alpha \in V(G)} \{\alpha\}) \sqcup (\bigsqcup_{e \in E(G)} \{e\})$ and then identify the point $I(e)$ with α if $I(e) = \alpha$ and $T(e)$ with α if $T(e) = \alpha$, for each $e \in E(G)$ and each $\alpha \in V(G)$; in this way we obtain a topological space G . Note that an orientation is naturally defined on each edge e through the homeomorphism $e \mapsto [0, 1]$.

An *embedding* of a graph G into the (Euclidean) plane is a one-to-one continuous mapping $f : G \rightarrow \mathbf{R}^2$. Not all graphs are embeddable into the plane; a graph which is embeddable into the plane is called *planar* (or *of genus 0*). An embedding f is said to be *smooth* if a curve $f(e)$ is smooth (except at endpoints) in the plane for each edge e . By a *plane graph* we shall mean a topological space $f(G)$ where $f : G \rightarrow \mathbf{R}^2$ is a smooth embedding of a graph G .

In what follows, since we shall fix a graph G and a smooth embedding f , we shall identify $f(G)$ with G ; thus a plane graph will be sometimes called a graph for short.

3. THE VDOVICHENKO FORMULA

The Vdovichenko formula for $S(\mathbf{x})$, in the generalized sense in [2], is described as follows.

Let G be a plane graph.

Let $\mathcal{I} = \{e \mid e \in E(G)\} \sqcup \{-e \mid e \in E(G)\}$ (disjoint union) be an index set and define operations on \mathcal{I} by

$$-m = -e, \quad I(m) = I(e), \quad T(m) = T(e), \quad \epsilon(m) = e \text{ if } m = e \text{ for some } e \in E(G);$$

$$-m = e, \quad I(m) = T(e), \quad T(m) = I(e), \quad \epsilon(m) = e \text{ if } m = -e \text{ for some } e \in E(G)$$

($m \in \mathcal{I}$). When $e \in E(G)$ we shall identify $e \in \mathcal{I}$ with a “move” on the edge e in the direction of the orientation assigned to it, and we shall identify $-e \in \mathcal{I}$ with a “move” on the edge in the opposite direction. Also, sometimes $m \in \mathcal{I}$ will be identified with a plane curve travelling from $I(m)$ to $T(m)$.

For a pair (m, m') of members in \mathcal{I} satisfying

$$(3) \quad m \neq -m' \quad \text{and} \quad T(m') = I(m)$$

define

$$(4) \quad W_{mm'} = x_{\epsilon(m')} \cdot \exp\left(\frac{i\theta_{mm'}}{2}\right), \quad \theta_{mm'} = \varphi_{mm'} + \int_{m'} \kappa ds$$

where $\varphi_{mm'}$ is a counterclockwise angle ($-\pi \leq \varphi_{mm'} \leq \pi$) a vector tangent to edges turns at the vertex $T(m')$ when moving from m' to m (When $\varphi_{mm'} \equiv \pi \pmod{2\pi}$, then $\varphi_{mm'} = \pi$ if m lies in the left side of m' , and $\varphi_{mm'} = -\pi$ otherwise), κ the geodesic curvature of the directed edge m' whose arc-length parameter is s .

Let $W_{mm'} = 0$ for pairs (m, m') which do not satisfy (3) and write

$$W = \left(W_{mm'} \mid m \in \mathcal{I}, m' \in \mathcal{I} \right).$$

The generating function $S(\mathbf{x})$ can be expressed in terms of this $2\#E(G) \times 2\#E(G)$ matrix.

The Vdovichenko formula.

$$(5) \quad S(\mathbf{x}) = \exp\left(-\sum_{p=1}^{\infty} \frac{1}{2p} \text{trace}(W^p)\right)$$

in the sense of formal power series in \mathbf{x} .

As was mentioned before there exists no rigorous proof; but it has been tested for many examples (see section 3.2 in [2] for example) and it is believed that the formula holds in general.

Since any square matrix can be transformed into an upper-triangle matrix by some similarity transformation, we have

Corollary.

$$(6) \quad S(\mathbf{x}) = \left[\det (I - W) \right]^{1/2}.$$

4. A LOOP EXPANSION FORMULA

Let us rewrite the formula (5) and consider its meaning more closely. To do this we introduce concepts of closed walks and loops on a graph.

A *walk* on G of length p ($p \geq 1$) is an ordered p -tuples (m_0, \dots, m_{p-1}) of members of \mathcal{I} which satisfies $T(m_i) = I(m_{i+1})$ for $i = 0, \dots, p-2$. A walk (m_0, \dots, m_{p-1}) is *closed* if $T(m_{p-1}) = I(m_0)$. For a closed walk $w = (m_0, \dots, m_{p-1})$ let $\mu(w)$ denote its *winding number* (in the sense of differential geometry; here we regard the closed walk as a piecewise smooth closed curve in the plane. See eq.(9)). We remark that the number ' $\mu(w) - 1 \bmod 2$ ' can be interpreted as the *number of intersection points of w modulo 2*. Thus define the sign of w , $\text{sgn}(w)$, to be $(-1)^{\mu(w)-1}$.

Let \mathbf{D}_p be the dihedral group of order $2p$ defined by the relations $S^p = R^2 = 1$ and $RSR^{-1} = S^{-1}$:

$$\mathbf{D}_p = \langle S, R \mid S^p = R^2 = 1, RSR^{-1} = S^{-1} \rangle.$$

\mathbf{D}_p consists of $2p$ elements of the form $S^j R^k$, $0 \leq j \leq p-1$, $k = 0, 1$. Define an action of \mathbf{D}_p on closed walks of length p by

$$\begin{aligned} S((m_0, \dots, m_{p-1})) &= (m_1, \dots, m_{p-1}, m_0), \\ R((m_0, \dots, m_{p-1})) &= (-m_{p-1}, \dots, -m_0). \end{aligned}$$

For a closed walk w of length p let

$$q(w) = \min\{j \mid 1 \leq j \leq p, S^j(w) = w\}, \quad d(w) = p/q(w);$$

the numbers $q(w)$ and $d(w)$ are said to be the *period* and the *degeneracy* of w respectively. In other words, if $\mathbf{C}_p = \langle S \mid S^p = 1 \rangle$ denotes the cyclic group of order p which is a subgroup of \mathbf{D}_p and if $\mathbf{C}_{p,w} = \{X \in \mathbf{C}_p \mid X(w) = w\}$ denotes the stabilizer of w under the action of \mathbf{C}_p , then $d(w) = \#\mathbf{C}_{p,w}$ and $q(w) = \#\mathbf{C}_p / \#\mathbf{C}_{p,w}$. Write

$$\mathbf{x}^w = \prod_{i=1}^p x_{\epsilon(m_i)}$$

(the set of variables \mathbf{x} was introduced in (1)). We note that for any closed walk w and any $X \in \mathbf{D}_p$ hold

$$(7) \quad \begin{aligned} \mu(X(w)) &\equiv \mu(w) \pmod{2}, \\ \text{sgn}(X(w)) &= \text{sgn}(w), \\ d(X(w)) &= d(w), \\ \mathbf{x}^{X(w)} &= \mathbf{x}^w. \end{aligned}$$

A closed walk (m_0, \dots, m_{p-1}) is said to be *allowed* if $m_{i+1} \neq -m_i$, $i = 0, \dots, p-2$, and $m_0 \neq -m_{p-1}$ (In other words, an allowed closed walk is a closed walk which contains no about-face turns). Let $\mathcal{W}_p(G)$ denote the set of all allowed closed walks of length p . The group \mathbf{D}_p acts on $\mathcal{W}_p(G)$. We note that for $w \in \mathcal{W}_p(G)$ the stabilizer $\mathbf{D}_{p,w} = \{X \in \mathbf{D}_p \mid X(w) = w\}$ coincides with $\mathbf{C}_{p,w}$ [Proof. Let $w = (m_0, \dots, m_{p-1}) \in \mathcal{W}_p(G)$ and suppose that $S^j R(w) = w$ for some j , $0 \leq j \leq p-1$; then we have $-m_{p-j-1-k} = m_k$, $0 \leq k \leq p-j-1$. If $p-j$ is odd we have $-m_i = m_i$ for some i , a contradiction. Otherwise we have $-m_{i+1} = m_i$ for some i ; again impossible since w is assumed to be allowed.]; hence $d(w) = \#\mathbf{D}_{p,w}$ and $2q(w) = \#\mathbf{D}_p / \#\mathbf{D}_{p,w}$.

An orbit of the action of \mathbf{D}_p on $\mathcal{W}_p(G)$ is called a *loop* of length p . Let $\mathcal{L}_p(G)$ denote the set of all loops of length p . $\mathcal{L}_p(G) = \mathbf{D}_p \backslash \mathcal{W}_p(G)$. By virtue of (7) the following definitions make sense for $L \in \mathcal{L}_p(G)$:

$$\text{sgn}(L) = \text{sgn}(w), \quad d(L) = d(w), \quad \mathbf{x}^L = \mathbf{x}^w$$

where $w \in L$. Now let us define the *weight* of L to be

$$\omega(L) = \text{sgn}(L) \cdot \frac{1}{d(L)}$$

The following lemma clarifies the meaning of $\text{trace } W^p$ in the Vdovichenko formula.

Lemma 1.

$$(8) \quad \sum_{L \in \mathcal{L}_p(G)} \omega(L) \mathbf{x}^L = -\frac{1}{2p} \text{trace}(W^p).$$

Write $W(w) = W_{m_{p-1}, m_{p-2}} \cdots W_{m_1, m_0} W_{m_0, m_{p-1}}$ for $w = (m_0, \dots, m_{p-1}) \in \mathcal{W}_p(G)$. Then

$$\text{trace}(W^p) = \sum_{w \in \mathcal{W}_p(G)} W(w).$$

For an allowed closed walk $w = (m_0, \dots, m_{p-1})$ we have an explicit expression for the winding number

$$(9) \quad \theta_{m_{p-1}, m_{p-2}} + \cdots + \theta_{m_1, m_0} + \theta_{m_0, m_{p-1}} = 2\pi\mu(w)$$

and hence

$$W(w) = \exp(i\pi\mu(w)) \cdot \mathbf{x}^w = -\text{sgn}(w) \cdot \mathbf{x}^w.$$

Thus $W(X(w)) = W(w)$ for $w \in \mathcal{W}_p(G)$ and $X \in \mathbf{D}_p$ (because of (7)), and therefore $W([w]) = W(w)$ is well-defined (where $[w] \in \mathcal{L}_p(G)$). Since the number of distinct representatives (allowed closed walks) of a loop $L = [w] \in \mathcal{L}_p(G)$ ($w \in \mathcal{W}_p(G)$) is

$$\#L = \frac{\#\mathbf{D}_p}{\#\mathbf{D}_{p,w}} = \frac{2p}{d(w)} = \frac{2p}{d(L)}$$

we have

$$\begin{aligned} \sum_{w \in \mathcal{W}_p(G)} W(w) &= \sum_{L \in \mathcal{L}_p(G)} \frac{2p}{d(L)} \cdot W(L) \\ &= - \sum_{L \in \mathcal{L}_p(G)} \frac{2p}{d(L)} \cdot \operatorname{sgn}(L) \cdot \mathbf{x}^L = -2p \sum_{L \in \mathcal{L}_p(G)} \omega(L) \mathbf{x}^L. \end{aligned}$$

Thus (8) is proved. \square

Thus we have the following exponential relationship for $S(\mathbf{x})$ and a summation over all loops.

Theorem 1. *Let G be a plane graph and let $S(\mathbf{x})$ be the generating function (1) in $\#E(G)$ variables $\mathbf{x} = \{x_e \mid e \in E(G)\}$. The following is equivalent to the Vdovichenko formula (5):*

$$(10) \quad S(\mathbf{x}) = \exp \left(\sum_{p=1}^{\infty} \sum_{L \in \mathcal{L}_p(G)} \omega(L) \mathbf{x}^L \right).$$

Proof. An immediate consequence of Lemma 1. \square

5. SOME LEMMAS PARTLY JUSTIFYING THE FORMULA

Although the author has not completed a proof of the formula (10) or (5), there are many pieces of “evidence” each of which would form a part of a complete proof. We describe a few of them.

5.1. Products of loops. Let $\mathcal{L}(G) = \cup_{p=1}^{\infty} \mathcal{L}_p(G)$ and refer to it as the set of all loops. For loops $L_1, L_2 \in \mathcal{L}(G)$ we consider a formal commutative product $L_1 L_2$ and define

$$\operatorname{sgn}(L_1 L_2) = \operatorname{sgn}(L_1) \operatorname{sgn}(L_2), \quad \mathbf{x}^{L_1 L_2} = \mathbf{x}^{L_1} \mathbf{x}^{L_2}.$$

As usual we will write $L^2 = LL$, etc.

Let us introduce the “empty loop” \emptyset which plays a role of unit element: i.e., $\emptyset L = L \emptyset = L$, $L \in \mathcal{L}(G)$. Define $\operatorname{sgn}(\emptyset) = 1$ and $\mathbf{x}^{\emptyset} = 1$.

Let $\mathcal{Z}(G)$ denote the set which consists of all products of loops and the empty loop:

$$\mathcal{Z}(G) = \{ L_1 \cdots L_k \mid k \geq 0, L_i \in \mathcal{L}(G) (i = 1, \dots, k) \}.$$

Each element $Z \in \mathcal{Z}(G)$ is factorized as

$$Z = L_1^{j_1} \cdots L_k^{j_k}$$

where the factors L_1, \dots, L_k are distinct loops and $j_1 \geq 1, \dots, j_k \geq 1$. The factorization is unique apart from the order of the factors.

When

$$Z = L_1^{j_1} \cdots L_k^{j_k}, \quad Z' = L_1^{j'_1} \cdots L_k^{j'_k};$$

$0 \leq j'_1 \leq j_1, \dots, 0 \leq j'_k \leq j_k$, we shall say that Z' is a divisor of Z . If Z and Z' have no common divisor but the empty loop \emptyset they are said to be relatively prime.

Lemma 2. Define $U(Z)$, $Z \in \mathcal{Z}(G)$, by

$$(11) \quad \exp \left(\sum_{L \in \mathcal{L}(G)} \omega(L) \mathbf{x}^L \right) = \sum_{Z \in \mathcal{Z}(G)} U(Z) \mathbf{x}^Z.$$

The coefficients $U(Z)$ have the following properties:

(a) $U(\emptyset) = 1$ and $U(L) = \omega(L) = \text{sgn}(L)/d(L)$ for $L \in \mathcal{L}(G)$, and in general

$$\begin{aligned} U(L_1^{j_1} \cdots L_k^{j_k}) &= \frac{1}{j_1! \cdots j_k!} \omega(L_1)^{j_1} \cdots \omega(L_k)^{j_k} \\ &= \frac{1}{j_1! \cdots j_k!} \frac{\text{sgn}(L_1)^{j_1} \cdots \text{sgn}(L_k)^{j_k}}{d(L_1)^{j_1} \cdots d(L_k)^{j_k}} \end{aligned}$$

where L_1, \dots, L_k are distinct loops and j_1, \dots, j_k nonnegative integers.

(b) If Z and Z' in $\mathcal{Z}(G)$ are relatively prime then

$$U(ZZ') = U(Z)U(Z').$$

(c) Let L_1, \dots, L_k be distinct loops and let $j_1 \geq 1, \dots, j_k \geq 1$. Then

$$|U(L_1^{j_1} \cdots L_k^{j_k})| = 1$$

if and only if $j_1 = \cdots = j_k = 1$ and $d(L_1) = \cdots = d(L_k) = 1$.

Proof. The definition of $U(Z)$ implies (a). (b) and (c) are corollaries of (a). \square

5.2. Sum of the coefficients corresponding to an Euler subgraph is

1. Let $\mathbf{x} = \{x_e \mid e \in E(G)\}$. Define a mapping $\mathbf{x} : \mathcal{Z}(G) \rightarrow \{\text{monomials}\}$ by $\mathbf{x}(Z) = \mathbf{x}^Z$. It is known that any Euler graph can be partitioned into loops. Thus, for each Euler subgraph $P \in \mathcal{P}(G)$ there exists a product of loops $Z \in \mathcal{Z}(G)$ such that $\prod_{e \in E(P)} x_e = \mathbf{x}^Z$. Let \mathcal{Z}_P denote the set of all such Z ; namely, for each $P \in \mathcal{P}(G)$ define

$$\mathcal{Z}_P = \left\{ Z \in \mathcal{Z}(G) \mid \prod_{e \in E(P)} x_e = \mathbf{x}^Z \right\}.$$

This is an easy lemma:

Lemma 3. *Let $P \in \mathcal{P}(G)$ be an Euler subgraph. Then, $\sum_{Z \in \mathcal{Z}_P} U(Z) = 1$.*

Proof. Let $1, \dots, k$ be the vertices of P whose degree $\deg_P \geq 4$. Write $2n_j = \deg_P(j)$, $1 \leq j \leq k$. At each vertex j there are $2n_j$ incident edges; the number of ways of joining two in pairs is $(2n_j - 1)!!$, of which $((2n_j - 1)!! + 1)/2$ ways have even transversal intersections (in a neighborhood of the vertex) and remaining $((2n_j - 1)!! - 1)/2$ ways have odd transversal intersections. Thus the number of ways of partitioning the Euler subgraph P into loops is $M_P = (2n_1 - 1)!! \times \dots \times (2n_k - 1)!!$ [that is, $\#\mathcal{Z}_P = \prod_{j=1}^k (2n_j - 1)!!$]; among those partitions $(M_P + 1)/2$ ways correspond to products of loops having even intersections (i.e., $\text{sgn}(Z) = 1$), and remaining $(M_P - 1)/2$ ways to products of loops having odd intersections (i.e., $\text{sgn}(Z) = -1$). Thus $\sum_{Z \in \mathcal{Z}_P} U(Z) = (M_P + 1)/2 - (M_P - 1)/2 = 1$. \square

5.3. Cancellation of terms which do not correspond to Euler subgraphs.

This is the most nontrivial part. The author has not completed justification for the title of this subsection. Only partial results are exhibited here.

For a loop $L \in \mathcal{L}_q(G)$ and a positive integer n , take an allowed closed walk $w = (m_0, \dots, m_{q-1}) \in \mathcal{W}_q(G)$ such that $[w] = L$, and write

$$w^{*n} = \underbrace{(m_0, \dots, m_{q-1}, \dots, m_0, \dots, m_{q-1})}_{\text{pattern } m_0, \dots, m_{q-1} \text{ repeats } n \text{ times}} \in \mathcal{W}_{nq}(G),$$

and define a loop $L^{*n} \in \mathcal{L}_{nq}(G)$ to be $L^{*n} = [w^{*n}]$. Clearly $L^{*1} = L$, and $\mathbf{x}^{L^{*n}} = (\mathbf{x}^L)^n$. Each loop L has a representation $L = L_1^{*d}$ for some $L_1 \in \mathcal{L}(G)$ where $d = d(L)$.

When $d(L) \geq 2$ the loop L is said to be *degenerate*.

Lemma 4. *Let L be a nondegenerate loop ($d(L) = 1$). Let $k \geq 1$. Then*

$$\text{sgn}(L^{*k}) = -(-\text{sgn}(L))^k, \quad d(L^{*k}) = k$$

and hence

$$U(L^{*k}) = -(-\text{sgn}(L))^k \frac{1}{k}.$$

Proof. Take an allowed closed walk w such that $[w] = L$, and let $\mu = \mu(w)$ be its winding number; thus “the number of intersection points mod 2” of w is $\nu = \mu - 1$. The degenerate walk w^{*k} has the winding number $\mu(w^{*k}) = k\mu$. Thus its “number of intersection points mod 2” is

$$\mu(w^{*k}) - 1 = k\mu - 1 = k\nu + k - 1 \equiv \begin{cases} k - 1 \pmod{2} & (\text{if } \nu \text{ is even}), \\ 1 \pmod{2} & (\text{if } \nu \text{ is odd}). \end{cases}$$

Since $\text{sgn}(L) = (-1)^{\mu(w^{*k})-1}$ the equality for sign is proved.

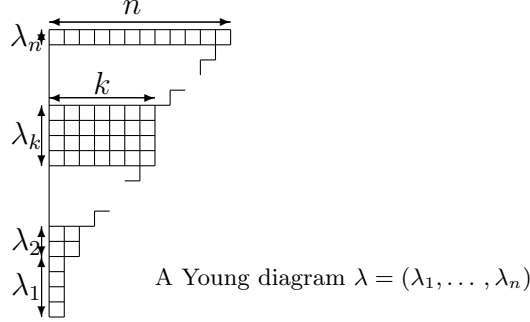
The statements on d and U are trivial. \square

The following nice cancellation occurs.

Lemma 5. *Let $Z_1 \in \mathcal{Z}(G)$ and let $L \in \mathcal{L}(G)$. Suppose that L is nondegenerate ($d(L) = 1$) and that L is not a divisor of Z_1 . Let $n \geq 2$. Then*

$$\sum_{\lambda \vdash n} U(Z_1 L^{\lambda_1} (L^{*2})^{\lambda_2} \dots (L^{*n})^{\lambda_n}) = 0.$$

Here “ $\lambda \vdash n$ ” means that $\lambda = (\lambda_1, \dots, \lambda_n)$ is an n -tuple of nonnegative integers such that $\sum_{k=1}^n k\lambda_k = n$ (see Figure). The summation is taken over all such λ .



Proof. Because of Lemma 2 (b) it is sufficient to prove the statement for $Z_1 = \emptyset$.

Let $\lambda \vdash n$ and put $Z_\lambda = \prod_{k=1}^n (L^{*k})^{\lambda_k}$. Using lemmas prepared above and $\sum_{k=1}^n k\lambda_k = n$ we have

$$\begin{aligned} U(Z_\lambda) &= U\left(\prod_{k=1}^n (L^{*k})^{\lambda_k}\right) = \prod_{k=1}^n U((L^{*k})^{\lambda_k}) \\ &= \prod_{k=1}^n \frac{1}{\lambda_k!} \operatorname{sgn}(L^{*k})^{\lambda_k} \frac{1}{d(L^{*k})^{\lambda_k}} \\ &= \prod_{k=1}^n \frac{1}{\lambda_k!} (-1)^{\lambda_k} (-\operatorname{sgn}(L))^{k\lambda_k} \frac{1}{k^{\lambda_k}} \\ &= (-1)^{\sum \lambda_k} (-\operatorname{sgn}(L))^{\sum k\lambda_k} \prod_{k=1}^n \frac{1}{\lambda_k! k^{\lambda_k}} \\ &= (-\operatorname{sgn}(L))^n (-1)^{\sum \lambda_k} \prod_{k=1}^n \frac{1}{\lambda_k! k^{\lambda_k}} \end{aligned}$$

and hence

$$\sum_{\lambda \vdash n} U(Z_\lambda) = (-\operatorname{sgn}(L))^n \sum_{\lambda \vdash n} (-1)^{\sum \lambda_k} \prod_{k=1}^n \frac{1}{\lambda_k! k^{\lambda_k}}.$$

Now recall a well-known fact that the number of elements of \mathbf{S}_n , the symmetric group of degree n , of type $\lambda = (\lambda_1, \dots, \lambda_n)$, i.e., having λ_k cycles of length k , $k = 1, 2, \dots, n$, is

$$h(\lambda) = \prod_{k=1}^n \frac{n!}{\lambda_k! k^{\lambda_k}}$$

(Cauchy's formula). The sign of a permutation of this type λ is

$$\prod_{k=1}^n \{(-1)^{k+1}\}^{\lambda_k} = (-1)^n (-1)^{\sum \lambda_k}.$$

Putting all this together we have

$$\sum_{\lambda \vdash n} U(Z_\lambda) = \frac{(\operatorname{sgn}(L))^n}{n!} \sum_{\lambda \vdash n} (-1)^n (-1)^{\sum \lambda_k} h(\lambda) = \frac{(\operatorname{sgn}(L))^n}{n!} \sum_{\sigma \in \mathbf{S}_n} \operatorname{sgn} \sigma.$$

This equals 0. □

Thus, what remains to be justified in order to complete a proof of the loop expansion formula (10) or the Vdovichenko formula (5) is the following claim. Here, for $L \in \mathcal{L}(G)$ and $n \geq 2$, the product $L^n = L \cdots L$ (n times) is said to be *duplicate loops*.

Claim. *Let \mathcal{Z}' be the set of all elements Z of $\mathcal{Z}(G)$ such that*

Z has no duplicate loops nor degenerate loops in factors

and that

*there exists at least one $e \in E(G)$ such that the
degree of \mathbf{x}^Z in the variable x_e is ≥ 2 .*

Then $\sum_{Z \in \mathcal{Z}'} U(Z) \mathbf{x}^Z = 0$.

REFERENCES

- [1] N. V. Vdovichenko, *A calculation of the partition function for a plane dipole lattice*, Zh. Eksp. Teor. Fys. **47** (1964), 715–719. (Soviet Phys. JETP **20** (1965), 477–479.)
- [2] M. Idzumi, *A generalization of Vdovichenko's method for Ising models on torus graphs*, J. Phys. A: Math. Gen. **32** (1999), 6915–6925.

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, SHIMANE UNIVERSITY, MATSUE 690-8504 JAPAN

E-mail address: idzumi@edu.shimane-u.ac.jp