Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
**31** (1998), pp. 27–55

# BUNDLE MAP THEORY IN THE CATEGORY OF WEAK HAUSDORFF k-SPACES

Hideaki Ōshima and Kouzou Tsukiyama

(Received December 10, 1997)

Abstract. We generalize Theorem 2.1 of H. Ōshima and K. Tsukiyama [12] and prove Theorem 2.3 of I. M. James [7] without his condition. We identify the boundary homomorphism of the homotopy exact sequence of the evaluation fibration with the generalized Whitehead product. The new point is that we work not in the category of $CW$-complexes but in the category of well based weak Hausdorff $k$-spaces.

## §0. Introduction

In §1 we introduce weak Hausdorff $k$-spaces. After studying numerable principal bundles in §2, we prove the function $\Phi : map^G(E, E') \to map(B, B')$ is continuous in §3, if spaces involved have the $k$-ification of the compact-open topology. Here $map^G(E, E')$ is the set of $G$-equivariant maps from a numerable principal $G$-space $E$ to another $E'$, $map(B, B')$ is the set of maps from $B$ to $B'$ and $\Phi$ assigns the induced map on base spaces to each $G$-map. At the end of this section we give a reasonable topology on $map^G(E, E')$ which makes $\Phi$ continuous with respect to the compact-open topology of $map(B, B')$ and whose $k$-ification is the same as that of the $k$-ification of the compact-open topology of $map^G(E, E')$. In §4, by using of A. Dold [2] we prove that the $k$-ification of the space $map^G(E, EG)$ having the compact-open topology is contractible, where $EG$ is the total space of a universal principal $G$-bundle. D. H. Gottlieb [5] proved this by a different method. We generalize Theorem 2.1 of H. Ōshima and K. Tsukiyama [12] and prove Theorem 2.3 of I. M. James [7] without his condition. In §5 we identify the boundary homomorphism of the homotopy exact sequence of the evaluation fibration with the generalized Whitehead product(see G. E. Lang [9]). The new

---

1991 *Mathematics Subject Classification.* 55R05, 55R10.
*Key words and phrases.* bundle map theory, fibration map theory, k-space.

point is that we work not in the category of $CW$- complexes but in the category of well based weak Hausdorff $k$-spaces. As a Corollary we generalize Theorem 2.2 of H. Ōshima and K. Tsukiyama [12]. In §6 we generalize Theorem 2.1 of I. M. James [7] to a theorem in the category of $\mathcal{F}$-spaces of J. P. May [10]. In §7 we slightly generalize the fibration map theory of [16](cf. [3], [4]).

## §1. Weak Hausdorff $k$-spaces

The papers of [11], [14] and [15] show that why it is convenient to work in the category of weak Hausdorff $k$-spaces. We review this notion briefly. In this note compact means quasi-compact and Hausdorff, and a map is a continuous function(see Remark in §7).

A subset $A$ of the topological space $X$ is compactly closed if $f^{-1}(A)$ is closed for all compact spaces $K$ and all maps $f : K \to X$. $U$ is a regular open subset of a space $X$ if each point of $U$ has a neighborhood whose closure is contained in $U$.

Let us call a space $X$ is a $k$-space if all compactly closed subsets are closed.

A space $X$ is weak Hausdorff if $f(K)$ is closed for all compact spaces $K$ and all maps $f : K \to X$. We let $WH$ be the category of weak Hausdorff spaces and maps, and let $WHK(Top)$ be the category of weak Hausdorff k-spaces and maps (topological spaces and maps). And $WHK_*(Top_*)$ will denote the well-based category of the respective one. In particular, if $X$ is a topological space, let $k(X)$ denote the space whose underlying set is that of $X$ and whose closed sets are compactly closed subsets of $X$.

If $f : X \to Y$ is a function, we let $k(f) : k(X) \to k(Y)$ be the same function. Then $k$ is a retraction functor from the category of all topological spaces onto the category of $k$-spaces. The weak Hausdorff property holds under the functor $k$.

Let $A_r$ be the space given by a subset $A$ of the space $X$ with the relative topology. And if $A$ is a subset of a $k$-space $X$, then the space $k(A_r)$ is called the $k$-subspace of $X$. If $X$ and $Y$ are spaces, let $X \otimes Y = k(X \times Y)$, where $\times$ denotes the usual Cartesian product. It is easy to see that the quotient space of a $k$- space is a $k$-space. For pairs of spaces $(X, K), (Y, U)$ we denote by $W(K, U)$ the set of maps $f : X \to Y$ with $f(K) \subset U$. We consider $map(X, Y)$ as a topological space having the compact open topology, that is, the topology generated by the family of subsets $W(K, U)$ for all compact subsets $K$ of $X$ and all open subsets $U$ of $Y$, unless otherwise stated. We set $Map(X, Y) = k(map(X, Y))$. The following lemma is used in §4.

**Lemma 1.1.** *If $u : A \to X$ and $v : A \to Y$ are maps in $WHK$, then the double mapping cylinder $M(u, v)$ is a weak Hausdorff $k$-space. In particular the mapping cylinder $M(u) = M(u, id_A)$ is a weak Hausdorff $k$-space.*

*Proof.* Recall that $M(u, v)$ is the quotient space of $I \times A + X + Y$ with respect to the relations

$$(0, a) \sim u(a), (1, a) \sim v(a) (a \in A).$$

Let $p : I \times A + X + Y \to M(u,v)$ be the projection. Since $I \times A + X + Y$ is a weak Hausdorff $k$-space, it suffices by [11, 2.4], to show that $(p \times p)^{-1}(\triangle M(u,v))$ is compactly closed. We have

$$(p \times p)^{-1}(\triangle_{M(u,v)}) = \{\triangle_{I\times A} \cup Z\} + \triangle_X + \triangle_Y + \\ Im(u_1) + Im(u_2) + Im(v_1) + Im(v_2)\},$$

where $Z$ is the union of $\cup[\{0\} \times u^{-1}(x) \times \{0\} \times u^{-1}(x) \mid x \in X]$ and $[\{1\} \times v^{-1}(y) \times \{1\} \times v^{-1}(y) \mid y \in Y]$, and $u_1 : A \to I \times A \times X, u_2 : A \to X \times I \times A, v_1 : A \to I \times A \times Y, v_2 : A \to Y \times I \times A$ are defined by $u_1(a) = (0, a, u(a)), u_2(a) = (u(a), 0, a), v_1(a) = (1, a, v(a)), v_2(a) = (v(a), 1, a)$. It follows from [11, 2.3]that $\triangle_{I\times A}, \triangle_X, Im(u_i)$ and $Im(v_i)$ are compactly closed. Set $u' = id \times u : I \times A \to I \times X$ and $v' = id \times v : I \times A \to I \times Y$. Then $(u' \times u')^{-1}(\triangle_{I\times X})$ and $(v' \times v')^{-1}(\triangle_{I\times Y})$ are compactly closed and hence so is $Z = [\{0\} \times A \times \{0\} \times A \cap (u' \times u')^{-1}(\triangle_{I\times X})] \cup [\{1\} \times A \times \{1\} \times A \cap (v' \times v')^{-1}(\triangle_{I\times Y})]$. Therefore $(p \times p)^{-1}(\triangle_{M(u,v)})$ is compactly closed. □

## §2. G-spaces

A topological group $G$ in $WHK$ is a weak Hausdorff $k$-space equipped with group structure such that the multiplication $G \otimes G \to G$ and the inversion $G \to G$ are continuous. We note that if $G$ is a weak Hausdorff topological group in $Top$, then $k(G)$ is a topological group in $WHK$. If $G$ is not assumed to be a $k$-space, then the left or right translation by $g \in G$ is continuous as a function $k(G) \to G$ but not as a function $G \to G$. An(right) action of the topological group $G$ in $WHK$ on the space $X$ in $WHK$ is a map $\mu : X \otimes G \to X$ such that $xe = x$ and $(xg_1)g_2 = x(g_1g_2)$, where $xg = \mu(x, g)$, for each $x \in X$, $g_1$, $g_2 \in G$($e$ is the unit of $G$). A $G$-space is a space in $WHK$ with an action of $G$ in $WHK$.

**Definition 2.1.** *A $G$-bundle is a triple $(E, p, B)$ of which $E$ is a $G$-space and $p : E \to B$ is a map such that $p = fp_E$, where $p_E : E \to E/G$ is an orbit map and $f : E/G \to B$ is a homeomorphism. The $G$-space $E$, with the given map $p$, is called a $G$-bundle over $B$ and the space $B$ is called the the base space of the bundle.*

A $G$-subspace $A$ of a $G$-space $X$ is a $G$-invariant subset $A$ of $X$ with the topology $k(A_r)$ and the $G$-action $k(A_r) \otimes G = A_r \otimes G \to k(A_r)$. Let $(X, p, B)$ and $(X', p', B')$ be $G$-bundles and let $u : X \to X'$ be a $G$-map. Then there exists a unique map $f : B \to B'$ such that $fp = p'u$. The map $f$ is called the induced map of $u$ and the pair $(u, f)$ is called a $G$-bundle map. If $B = B'$ and if $f = id_B$, then $u$ is called a $G$-map over $B$.

**Definition 2.2.** *A $G$-space $X$ is called nice if $\{(x, xg) \mid x \in X, g \in G\}$ is closed in $X \otimes X$.*

We note that a $G$-space $X$ is nice if and only if $X/G \in WHK$ ([11, 2.3]).

Let X be a G-space. We set

$$X^{*'} = \{(x, xg) \in X \times X \mid x \in X, g \in G\} \subset X \otimes X,$$
$$X^* = k(X^{*'}).$$

The function $\tau : X^* \to G$ defined by $\tau(x, xg) = g$ is called the translation function. For every $x \in X$, the function $\kappa(x) : G \to k(xG)$ defined by $\kappa(x)(g) = xg$ is called the admissible map for $X$ with respect to $x$. Note that the admissible map is a continuous bijection. Note also that if $X$ is nice, then $X^{*'} = (p_X \otimes p_X)^{-1}(\triangle_{X/G})$ is closed in $X \otimes X$. Hence $X^* = X^{*'}$.

**Definition 2.3.** *A $G$-space $X$ is called principal provided it is a nice and free $G$-space with a continuous translation function $\tau : X^* \to G$. A principal $G$-bundle over $B$ is a $G$-bundle $X$ over $B$, where $X$ is a principal $G$-space.*

It is easily shown that every $G$-subspace of a principal $G$-space is principal.

**Proposition 2.4.** *If $(X, p, B)$ is a principal $G$-bundle, then every admissible map is a homeomorphism $\kappa(x) : G \to p^{-1}(p(x))$.*

*Proof.* Since $B$ is $T_1$, it follows that $p^{-1}(p(x)) = xG$ is closed and hence $xG \in WHK$. The inverse function of $\kappa(x)$ is

$$\tau(x, \ ) : p^{-1}(p(x)) \cong x \otimes p^{-1}(p(x)) \subset X^* \to G.$$

Since $\tau$ is continuous, so is $\tau(x, )$. Therefore $\kappa(x)$ is a homeomorphism. □

**Lemma 2.5.** *The product $G$-space $B \otimes G$, where the action of $G$ is given by $(b, g_1)g_2 = (b, g_1g_2)$, is principal whenever $B$ is weak Hausdorff.*

*Proof.* Let $p : B \otimes G \to B' = B \otimes G/G$ be the quotient map. There exists canonically a continuous bijection $f : B' \to k(B)$ such that $fp = q$, where $q : B \otimes G \to k(B)$ is the projection. Since $q$ is an open map, it follows that $f$ is a homeomorphism and hence $B' \in WHK$. Thus the $G$- space $B \otimes G$ is nice by [11, 2.3], and hence $(B \otimes G)^* = (B \otimes G)^{*'}$.

To prove the continuity of the translation function, observe that $((b, g), (b', g')) \in (B \otimes G)^*$ if and only if $b = b'$, and the translation function has the form $\tau((b, g), (b, g')) = g^{-1}g'$. Since $\tau$ factorizes as $(B \otimes G)^* \subset (B \otimes G) \otimes (B \otimes G) \to G \otimes G \to G$, where the middle function is the projection and the last function assigns to $(g, g')$ the element $g^{-1}g'$, it is continuous and hence so is $\tau$. This completes the proof. □

Notions of principal $G$-bundle map and an isomorphism are defined as usual.

**Definition 2.6.**

(1) *The product principal $G$- bundle over $B$ is the principal $G$-bundle $(B \otimes G, p, B)$, where $B \in WHK$ and $p$ is the projection on the first factor. A principal $G$-bundle over $B$ is called trivial if it is isomorphic to the product principal $G$-bundle over $B$.*

(2) *(2) A principal $G$-bundle $(X, p, B)$ is trivial over $B' \subset B$ if the principal $G$-bundle $k(p^{-1}B') \to (k(p^{-1}B'))/G$ is trivial.*

**Definition 2.7.** *A principal $G$-bundle $X$ over $B$ is numerable if there exists a locally finite partition of unity $\{\theta_j : B \to I\}$ such that the bundle is trivial over $\theta_j^{-1}(0,1]$ for each $j$.*

**Lemma 2.8.** *If $f : Z \to I$ is a map, then $f^{-1}(0,1]$ is a regular open subset of $Z$. If in addition $Z \in WHK$, then so is $f^{-1}(0,1]$.*

*Proof.* Let $z \in f^{-1}(0,1]$. Then $f^{-1}(f(z)/2,1]$ is a neighborhood of $z$, $f(Cl(f^{-1}(f(z)/2,1])) \subset [f(z)/2,1] \subset (0,1]$ and $Cl(f^{-1}(f(z)/2,1]) \subset f^{-1}(0,1]$, so $f^{-1}(0,1]$ is regular open. The second assertion follows from [14, 2.4]. □

By this lemma, we have the following by [14, 2.4]

**Proposition 2.9.** *Let $(X, p, B)$ be a principal $G$-bundle, and let $\theta : B \to I$ be a map. Then $p^{-1}\theta^{-1}(0,1]$ and $\theta^{-1}(0,1]$ are weak Hausdorff $k$-spaces, and $(p^{-1}\theta^{-1}(0,1])/G = \theta^{-1}(0,1]$.*

**Proposition 2.10.** *Let $(X, p, B)$ be a $G$-bundle, where $X$ is a free $G$-space, and let $\{\theta_j : B \to I\}$ be a set of maps such that $\cup_j \theta_j^{-1}(0,1] = B$ and the bundle is trivial over $\theta_j^{-1}(0,1]$ for every $j$. Then $X$ is principal.*

*Proof.* Since $p^{-1}\theta_j^{-1}(0,1]$ is a $k$-space by 2.8, and since $p : p^{-1}\theta_j^{-1}(0,1] \to \theta_j^{-1}(0,1]$ is proclusive, it follows that $\theta j^{-1}(0,1]$ is a $k$-space. Hence the triviality of the bundle over $\theta_j^{-1}(0,1]$ implies that $p^{-1}\theta_j^{-1}(0,1] \cong \theta_j^{-1}(0,1] \otimes G$. Thus $\theta_j^{-1}(0,1] \otimes G \in WHK$ and hence $\theta_j^{-1}(0,1] \in WHK$. Therefore the diagonal set $\triangle_j \subset \theta_j^{-1}(0,1] \otimes \theta_j^{-1}(0,1]$ is closed by [11, 2.3]. We want to prove that $X$ is nice, by showing $\triangle_B$ is closed in $B \otimes B$. Let $\triangle_I \subset I \otimes I$ be the diagonal set. Then $\triangle_B = \cup_j \triangle_j \subset \cup_j (\theta_j \otimes \theta_j)^{-1}(\triangle_I)$. Let $\{z_a\}$ be a directed family of points in $\triangle_B$ which converges to a point $z_0 \in B \otimes B$. Since $(\theta_j \otimes \theta_j)(z_a) \in \triangle_I$ and $\triangle_I$ is closed, it follows that $(\theta_j \otimes \theta_j)(z_0) \in \triangle_I$ for all $j$. Choose $j$ such that $(\theta_j \otimes \theta_j)(z_0) \in (0,1] \otimes (0,1]$. Then $z_0 \in \theta_j^{-1}(0,1] \otimes \theta_j^{-1}(0,1]$. Since the last space is an open subspace of $B \otimes B$ by 2.3, there exists $a_0$ such that $z_a \in \theta_j^{-1}(0,1] \otimes \theta_j^{-1}(0,1]$ for all $a \geq a_0$. Thus $\{z_a | a \geq a_0\}$ is a convergent directed family of points in the closed subset $\triangle_j$ of $\theta_j^{-1}(0,1] \otimes \theta_j^{-1}(0,1]$, so $z_0 \in Cl(\triangle_j) \cap \theta_j^{-1}(0,1] \otimes \theta_j^{-1}(0,1] = \triangle_j \subset \triangle_B$. Therefore $\triangle_B$ is closed. We must show that

$\tau : X^* \to G$ is continuous. Let $(x,y) \in X^*$. Choose $j$ with $p(x) \in \theta_j^{-1}(0,1]$. Since $p^{-1}\theta_j^{-1}(0,1]$ is regular open by 2.8, it follows easily that $p^{-1}\theta_j^{-1}(0,1] \otimes p^{-1}\theta_j^{-1}(0,1]$ is an open neighborhood of (x,y) in $X \otimes X$. Since $p^{-1}\theta_j^{-1}(0,1] \cong \theta_j^{-1}(0,1] \otimes G$ and since $\theta_j^{-1}(0,1]$ is weak Hausdorff, $\tau$ is continuous on $(p^{-1}\theta_j^{-1}(0,1])^{*'}$ by 2.5, so $\tau$ is continuous at $(x,y)$ as a function $X^{*'} \to G$, and $\tau$ is continuous as a function $X^* \to G$. $\square$

**Proposition 2.11.** *Let $(X,p,B)$ be a $G$-bundle and let $f : B_1 \to B$ be a map. We set*

$$X_1 = \{(b_1, x) \in B_1 \times X \mid f(b_1) = p(x)\}$$

*and let $p_1 : X_1 \to B_1$ and $f_1 : X_1 \to X$ be the projections.*

(1) *The spaces $X_1$ and $k(X_1)$ have natural $G$-actions for which $f_1$ is a $G$-map and there is a homeomorphism $h : X_1/G \to B_1$ making the following diagram commutative and there is a homeomorphism $h : X_1/G \to B_1$ making the following diagram commutative*

$$\begin{array}{ccccccc} k(X_1) & \xrightarrow{id} & X_1 & \xrightarrow{=} & X_1 & \xrightarrow{f_1} & X \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\ k(X_1)/G & \xrightarrow{id} & X_1/G & \xrightarrow{h} & B_1 & \xrightarrow{f} & B. \end{array}$$

*If $B_1 \in WH$, then $k(X_1)$ is a $G$-space.*

(2) *If $X$ is a nice $G$-space and $B_1 \in WH$, then $k(X_1)$ is also a nice $G$-space.*

(3) *If $X$ is a free $G$-space and $B_1 \in WH$, then $k(X_1)$ is also a free $G$-space.*

(4) *If $X$ is a free $G$-space having continuous translation function and if $B_1 \in WH$, then the translation function of $k(X_1)$ is also continuous.*

(5) *If $(X,p,B)$ is a numerable principal $G$-bundle and $B_1 \in WHK$, then $(k(X_1), p, k(X_1)/G)$ is also a numerable principal $G$-bundle and $id : k(X_1)/G \to X_1/G$ is a homeomorphism.*

*Proof.* We define the $G$-actions on $X_1$ and $k(X_1)$ by $(b_1,x)g = (b_1,xg)$. Since they are determined by maps

$$k(X_1) \otimes G \to X_1 \otimes G \xrightarrow{f_1 \otimes id} X \otimes G \xrightarrow{\pi} X$$

$$k(X_1) \otimes G \to X_1 \otimes G \xrightarrow{p_1 \otimes id} B_1 \otimes G \xrightarrow{\pi} B_1,$$

they are continuous, where $\pi$ are the projections. Obviously $f_1$ is a $G$-map. If $B_1 \in WH$, then $k(X_1) \in WHK$, which is a $G$-space. We define $h((b_1,x)G) = b_1$. Since $X_1/G$ has the quotient topology, the function $h$ is continuous. As is easily seen, $h$ is bijective. Note that $p$ is an open map. It follows from [6, p.1]that $p_1$ is an open map, and so is $h$, hence $h$ is a homeomorphism. This proves (1). To prove (2), suppose that $X$ is nice and $B_1 \in WH$. Then $\Omega = \{(x, xg) \mid x \in X, g \in G\}$ and $\triangle_B$ are closed in $X \otimes X$ and $B \otimes B = k(B) \otimes k(B)$ respectively. Thus $\{((b, x), (b, xg) \mid x \in X, g \in G\} = ((f_1 \circ id) \otimes (f_1 \circ id))^{-1}\Omega \cap (kp_1 \otimes kp_1)^{-1}\triangle_B$ is closed in $k(X_1) \otimes k(X_1)$. Therefore $k(X_1)$ is a nice $G$-space.

(3) is trivial. To prove (4), suppose that $X$ is a free $G$-space with continuous translation function and $B_1 \in WH$. Since the translation function $\tau_1$ for $k(X_1)$ is the composite of the maps

$$(k(X_1))^* \xrightarrow{r_*} X^* \xrightarrow{\tau} G,$$

where $r = f_1 \circ id : k(X_1) \to X$ is the projection and $\tau$ is the translation function of $X$, it follows that $\tau_1$ is continuous. This proves (4).

Suppose that $(X, p, B)$ is a numerable principal $G$-bundle and $B_1 \in WHK$. Then $k(X_1)$ is a principal $G$-space, by (2), (3) and (4). Let $\{\theta_j : B \to I\}$ be a partition of unity which defines the numerability of the bundle $(X, p, B)$. Then $\{\theta_j fh \circ id\}$ is a locally finite partition of unity on $k(X_1)/G$. Put $U_j = \pi_j^{-1}(0, 1], V_j = f^{-1}\pi_j^{-1}(0, 1], X_j = p_1^{-1}(V_j)$, and $p_j = p_1 \mid X_j : X_j \to V_j$. Then $\pi_j^{-1}(0, 1]$ and $f^{-1}\pi_j^{-1}(0, 1]$ are in $WHK$ by 2.8. Without loss of generality, we can assume that $p : p^{-1}(U_j) \to U_j$ is the product $G$-bundle. Then $X_j = \{(v, u, g) \in V_j \times (U_j \otimes G) \mid u = f(v)\}$ and $p_j(v, f(v), g) = v$. Let $t : V_j \otimes G \to X_j$ be defined by $t(v, g) = (v, f(v), g)$. Then $t$ is continuous and factorizes into $V_j \otimes G \to k(X_j) \to X_j$. As is easily seen, $t : V_j \otimes G \to k(X_j)$ is a homeomorphism. This implies that $p : k(X_1) \to k(X_1)/G$ is trivial over $(\pi_j fh \circ id)^{-1}(0, 1]$ if $id : k(X_1)/G \to X_1/G$ is a homeomorphism. Since $k(p_j)t : V_j \otimes G \cong k(X_j) \to V_j$ is the projection, $k(p_j)$ is proclusive. Consider the following commutative diagram

$$\begin{array}{ccc} \coprod_j k(X_j) & \longrightarrow & k(X_1) \\ {\scriptstyle \coprod k(p_j)}\downarrow & & \downarrow{\scriptstyle k(p_1)} \\ \coprod_j V_j & \longrightarrow & B_1. \end{array}$$

Since $\coprod k(p_j)$ and $i$ are proclusive, so is $k(p_1)$. By the commutative diagram

$$\begin{array}{ccc} k(X_1) & \xrightarrow{=} & k(X_1) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle k(p_1)} \\ k(X_1) & \xrightarrow{id} & X_1/G \cong B_1, \end{array}$$

it follows that $h \circ id$ is a homeomorphism, and so is $id : k(X_1)/G \to X_1/G$. Thus $p : k(X_1) \to k(X_1)/G$ is trivial over $(\pi_j fh \circ id)^{-1}(0, 1]$ for all $j$. This proves (5). □

Under the notations of 2.11, we define

**Definition 2.12.**

(1) *Let* $(X,p,B)$ *be a* $G$*-bundle,* $B_1 \in WH$*, and* $f : B_1 \to B$ *be a map. The* $G$*-space* $k(X_1)$ *is denoted by* $f^*X$ *and is called the induced* $G$*-space under* $f$*.*

(2) *Let* $\xi = (X,p,B)$ *be a numerable principal* $G$*-bundle and let* $f : B_1 \to B$ *be a map, where* $B_1 \in WHK$*. The bundle* $k(p_1) : f^*X = k(X_1) \to B_1$ *is called the induced* $G$*-bundle of* $\xi$ *under* $f$*.*

By 2.11(5), we have

**Proposition 2.13.** *Any induced* $G$*-bundle of a numerable principal* $G$*-bundle is numerable.*

## §3. Equivariant function spaces

In this section, we prove the function $\Phi : map^G(E,E') \to map(B,B')$ which is defined in 3.5, is continuous, if spaces involved have the $k$-ification of the compact-open topology.

For $G$-spaces $X$ and $Y$, we define functions

$$\psi, \psi_1, \psi_2 : map(X,Y) \times G \to map(X,Y)$$

by

$$\begin{aligned} \psi(f,g)(x) &= f(xg^{-1})g, \\ \psi_1(f,g)(x) &= f(x)g, \\ \psi_2(f,g)(x) &= f(xg^{-1}), \text{where } f \in map(X,Y). \end{aligned}$$

Then $\psi$ factorizes into $map(X,Y) \times G \to map(X,Y) \times G \to map(X,Y)$, where the first function assigns $(\psi_1(f,g),g)$ to $(f,g)$ and the second function is $\psi_2$.

**Lemma 3.1.** *For each* $g \in G, \psi(\ ,g) : map(X,Y) \to map(X,Y)$ *is a homeomorphism.*

*Proof.* Obviously $\psi(\psi(f,g),g') = \psi(f,gg')$ and $\psi(f,e) = f$, so $\psi(,g)$ is a bijection and $\psi(\ ,g)^{-1} = \psi(\ ,g^{-1})$. Similar relations hold for $\psi_i$. Thus it suffices to prove that $\psi_i(\ ,g) : map(X,Y) \to map(X,Y)$ are continuous.

Let $f \in map(X,Y)$ and first assume $\psi_1(f,g) \in W(K,U)$. Then $f \in W(K,Ug^{-1})$. If $f' \in W(K,Ug^{-1})$, then $\psi_1(f',g)(K) = f'(K)g \subset U$, so $\psi_1(W(K,Ug^{-1}),g) \subset W(K,U)$. If $U$ is open in $Y$, then so is $Ug^{-1}$. Thus this shows that $\psi_1(\ ,g)$ is continuous.

Let $f \in map(X,Y)$ and assume $\psi_2(f,g) \in W(K,U)$. Then $f \in W(Kg^{-1},U)$, and $\psi_2(W(Kg^{-1},U),g) \subset W(K,U)$. If $K$ is compact, then so is $Kg^{-1}$. Thus $\psi_2(\ ,g)$ is continuous. □

**Proposition 3.2.** *If $X$ and $Y$ are $G$ -spaces, then $Map(X,Y)$ is also a $G$-space by the actions $\psi$.*

*Proof.* Since $map(X,Y)\times G$ is weak Hausdorff, it suffices to show that $\psi_i$ is continuous on each compact set. We prove this by showing that $\psi_i$ is continuous on $map(X,Y)\times D$ for $D$ compact. Let $R_g : G\to G$ be the map $R_g(g')=g'g$. By the commutative square

$$\begin{array}{ccc} map(X,Y)\times D & \xrightarrow{\psi_i} & map(X,Y) \\ {\scriptstyle id\times R_g}\downarrow & & \downarrow{\scriptstyle \psi_i(\ ,g)} \\ map(X,Y)\times D_g & \xrightarrow{\psi_i} & map(X,Y), \end{array}$$

it suffices to show that $\psi_i$ is continuous at $(f,e)$ on $map(X,Y)\times D$ for each compact set $D$ with $e\in D$.

Assume $\psi_1(f,e)\in W(K,U)$. Since the action is continuous, the composition

$$Y\times D = Y\otimes D\subset Y\otimes G\to Y$$

is continuous, so for each $x\in K$ there exist open neighborhoods $V(f(x))$ of $e$ in $D$ and $V'(f(x))$ of $f(x)$ in $Y$ such that $V'(f(x))V(f(x))\subset U$. Since $f(K)$ is quasi compact, there exist $x_1,...,x_n$ in $K$ such that $\cup_i V'(f(x_i))\supset f(K)$. Let $V=\cap_i V(f(x_i))$ and $V'=\cup_i V'(f(x_i))$. Then $f\in W(K,V')$ and $\psi_1(W(K,V'),V)\subset W(K,U)$. This shows that $\psi_1$ is continuous at $(f,e)$ on $map(X,Y)\times D$.

Assume $\psi_2(f,e)\in W(K,U)$, that is, $f(K)\subset U$. By the similar methods, we see that there exist a compact neighborhood $V$ of $e$ in $D$ and a neighborhood $V'$ of $K$ in $X$ such that $V=V^{-1}$ and $f(V'V)\subset U$. It then follows that $f\in W(KV,U)$ and $\psi_2(W(KV,U),V)\subset W(K,U)$, so that $\psi_2$ is continuous at $(f,e)$ on $map(X,Y)\times D$. □

**Proposition 3.3.** *Let $X$ and $Y$ be $G$-spaces. Suppose one of the following two conditions. Then $\psi : map(X,Y)\times G\to map(X,Y)$ is continuous.*

(1) *$X$ has the trivial $G$-action and $Y$ is locally compact.*
(2) *$G$ is locally compact.*

*Proof.* Assume (1). Then $Y\times G = Y\otimes G$. By the same proof as above, $\psi_1=\psi$ is continuous at $(f,e)\in map(X,Y)\times G$, so $\psi_1$ is continuous at each point. Assume (2). Then $X\times G = X\otimes G$ and $\psi_1$ is continuous by (1). By the same proof as that of 3.2, $\psi_2$ is also continuous and hence so is $\psi$. □

**Definition 3.4.** *Let $X$ and $Y$ be $G$-spaces. We set*

$$\begin{aligned} map^G(X,Y) &= map(X,Y)^G \\ &= \{f \in map(X,Y) \mid \psi(f,g) = f \quad for\ all\ g \in G\}, and \\ Map^G(X,Y) &= k(map^G(X,Y)). \end{aligned}$$

We note that $Map^G(X,Y)$ is a fixed point set of the action in 3.2.

For any $G$-bundles $(X,p,B)$ and $(X',p',B')$, we have canonically the function

$$\Phi : map^G(X,X') \to map(B,B')$$

making the following square commutative for each $f \in map^G(X,X')$

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\Phi(f)} & B'. \end{array}$$

**Theorem 3.5.** *The function $\Phi : Map^G(X,X') \to Map(B,B')$ is continuous.*

*Proof.* We prove this by showing that $\Phi : map^G(X,X') \to map(B,B')$ is continuous on each compact set. Let $D$ be a compact set of $map^G(X,X')$, $f \in D$, and assume $\Phi(f) \in W(K,U)$ where $K$ is a compact set in $B$, and $U$ is open in $B'$.

Since $ev : Map(X,X') \otimes X \to X'$ is continuous by [15, 3.5], $ev$ is continuous on $D \times X = D \otimes X$. We can define uniquely a map $\Phi'$ which makes the following square commutative

$$\begin{array}{ccc} D \times X & \xrightarrow{ev} & X' \\ {\scriptstyle id\times p}\downarrow & & \downarrow{\scriptstyle p'} \\ D \times B & \xrightarrow{\Phi'} & B'. \end{array}$$

There exist neighborhoods $V$ of $f$ in $D$ and $V'$ of $K$ in $B$ such that $\Phi'(V,V') = \Phi(V)(V') \subset U$. In particular, if $v \in V$, then $\Phi(v)(K) \subset \Phi(V)(V') \subset U$, so $\Phi(V) \subset W(K,U)$, thus $\Phi$ is continuous at $f$ on $D$, hence $\Phi : Map^G(X,X') \to Map(B,B')$ is continuous. $\square$

**Proposition 3.6.** *Let $(X,p,B)$ and $(X',p',B')$ be $G$-bundles.*

(1) *If $G$ is compact and $X$ is locally compact, then $\Phi : map^G(X,X') \to map(B,B')$ is continuous.*
(2) *If (X,p,B) is a numerable principal $G$-bundle and $B$ is locally compact, then $\Phi : map^G(X,X') \to map(B,B')$ is continuous.*
(3) *If $X$ is locally compact, then $\Phi : map^G(X,X') \to map(B,B')$ is continuous.*
(4) *If $(X,p,B)$ is a numerable principal $G$-bundle such that $B$ is a $CW$-complex, then $\Phi : map^G(X,X') \to map(B,B')$ is continuous.*

*Proof.* Let $f \in map^G(X, X')$ and assume $\Phi(f) \in W(K, U)$, where $U$ is an open subset of $B'$ and $K$ is a compact subset of $B$.

First assume $G$ is compact. Then $p$ is a closed mapping by the proof [13, 1.1.1.]. By [1, 7.8(2)], $p^{-1}K$ is quasi-compact. Moreover if $X$ is locally compact Hausdorff and if $K$ is compact, then $p^{-1}K$ is compact by [1, 7.8(3)]. Since $\Phi W(p^{-1}K, p'^{-1}U) \subset W(K, U)$, this proves (1).

Assume that $(X, p, B)$ is a numerable principal $G$-bundle and that $B$ is locally compact. Let $a \in K$. Then there exist an open neighborhood $O_a$ of $a$ and a $G$-isomorphism over $O_a$

$$h_a : O_a \otimes G = O_a \times G \cong p^{-1}O_a.$$

Since $\Phi(f)^{-1}(U) \cap O_a$ is an open neighborhood of $a$, there exists a compact neighborhood $B_a$ of $a$ such that $B_a \subset \Phi(f)^{-1}(U) \cap O_a$. Since the interiors $\{Int(B_a) \mid a \in K\}$ covers the quasi-compact set $K$, there are finite points $a(1), ..., a(n)$ with $K \subset \cup_i Int(B_{a(i)})$. Put $D_i = h_{a(i)}(B_{a(i)} \times \{e\})$ and $D = \cup_i D_i$. Then $D$ is compact, $K \subset p(D) \subset \Phi(f)^{-1}(U), f \in W(D, p'^{-1}U)$, and $\Phi(W(D, p'^{-1}U)) \subset W(K, U)$. This proves (2).

Assume $X$ is locally compact. Let $x \in p^{-1}K$. Then there exists a compact neighborhood $L_x$ of $x$ with $f(L_x) \subset p'^{-1}U$. Since $\{p(Int(L_x)) \mid x \in p^{-1}K\}$ is an open covering of the quasi-compact set $K$, there are finite points $x(1), ..., x(n)$ such that $K \subset \cup_i p(Int(L_{x(i)})$. Put $L = \cup_i L_{x(i)}$. Then $L$ is compact by [11, 2.1] and $K \subset p(L), f(L) \subset p'^{-1}U$, so $f \in W(L, p'^{-1}U)$, and $\Phi(W(L, p'^{-1}U) \subset W(K, U)$. This proves (3).

Assume $(X, p, B)$ is a numerable principal $G$-bundle with $B$ a $CW$-complex. Let $f \in map^G(X, X')$ and $\Phi(f) \in W(K, U)$. Since $B$ is a $CW$-complex, $K$ is closed and there is a finite subcomplex $S$ containing $K$. Let $\{h_i : D^{n(i)} \to S\}$ be the set of characteristic maps of closed cells of $S$, where $D^n$ is the $n$-dimensional disk. Notice that $D^{n(i)} \times G = D^{n(i)} \otimes G$ by [14, 4.3]. We have then the following commutative diagram in which $h'_i$ is a $G$-map

$$\begin{array}{ccccc} D^{n(i)} \times G & \xrightarrow{h'_i} & X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ D^{n(i)} & \xrightarrow{h_i} & B & \xrightarrow{\Phi(f)} & B' \end{array}$$

Set $D_i = h'_i(D^{n(i)} \times \{e\})$ and $K' = (\cup_i D_i) \cap p^{-1}(K)$. Then $p(K') = K$, and $K'$ is compact, because $D_i$ is compact, by [11, 2.1]. Moreover $f \in W(K', p'^{-1}U)$ and $\Phi(W(K', p'^{-1}U) \cap map^G(X, X')) \subset W(K, U)$, so $\Phi : map^G(X, X') \to map(B, B')$ is continuous. $\square$

Sometimes it is convenient to give $map^G(X, X')$ a stronger topology as follows (see §6):

**Definition 3.7.** *Let $(X, p, B)$ and $(X', p', B')$ be $G$-bundles. We define*

$$\begin{aligned} \underline{map}^G(X, X') &= \{(f, h) \mid f \in map^G(X, X'), h \in map(B, B'), p'f = hp\}, \\ \Phi : \underline{map}^G(X, X') &\to map(B, B'), \Phi(f, h) = h, \\ \underline{Map}^G(X, X') &= k(\underline{map}^G(X, X')). \end{aligned}$$

**Lemma 3.8.** *Let $(X, p, B)$ and $(X', p', B')$ be $G$-bundles. Then*

(1) *$\Phi : \underline{map}^G(X, X') \to map(B, B')$ is continuous.*
(2) *The projection induces a continuous bijection*

$$\pi : \underline{map}^G(X, X') \to map^G(X, X')$$

*and a homeomorphism*

$$\pi : \underline{Map}^G(X, X') \cong Map^G(X, X').$$

*Proof.* (1) is trivial. The inverse of $\pi$ assigns $f$ into $(f, \Phi(f))$, and it is continuous on each compact set by the proof of 3.5. Thus by [14, 3.2], $\pi : \underline{Map}^G(X, X') \cong Map^G(X, X')$. □

## §4. Equivariant function spaces of principal bundles

In this section, by using of A. Dold [2] we prove that the $k$-ification of the space $map^G(E, EG)$ having the compact-open topology is contractible, where $EG$ is the total space of a universal principal $G$-bundle. D. H. Gottlieb [5] proved this by a different method. We generalize Theorem 2.1 of H. Ōshima and K. Tsukiyama [12] and prove Theorem 2.3 of I. M. James [7] without his condition.

From now on we assume that a topological group $G$ in $WHK$ is well based, that is, $(G, e)$ is an $NDR$-pair, unless otherwise stated. Then there is a universal principal $G$-bundle $(EG, p, BG)$ by [11, 9.l7].

Let $(X, p, B)$ and $(X', p', B')$ be numerable principal $G$-bundles and let $b : B \to BG$ and $b' : B' \to BG$ be classifying maps of them. We set

$$\begin{aligned} map(B, B'; b, b') &= \{h \in map(B, B') \mid b'h \simeq b\}, \\ Map(B, B'; b, b') &= k(map(B, B'; b, b')). \end{aligned}$$

Note that $\Phi : map^G(X, X') \to map(B, B')$ has the image $map(B, B'; b, b')$.

**Proposition 4.1.** *$\Phi : Map^G(X, X') \to Map(B, B')$ has the CHP(covering homotopy property) for all weak Hausdorff $k$-spaces (without the assumption that $(G, e)$ is an $NDR$-pair), and so is $\Phi : Map^G(X, X') \to Map(B, B'; b, b')$. They have the $CHEP$ (covering homotopy extension property) for $NDR$-pairs in $WHK$.*

*Proof.* The last assertion follows from the former and [8, 6.44]. Let $Z \in WHK$. Assume the following commutative diagram

$$\begin{array}{ccccc} \{0\} \otimes Z & \xrightarrow{h} & Map^G(X, X') & \subset & Map(X, X') \\ \cap & & \downarrow{\scriptstyle \Phi} & & \\ I \otimes Z & \xrightarrow{H} & Map(B, B'). & & \end{array}$$

Taking adjoints, we have the following commutative diagram by [15, 3.6]

$$\begin{array}{ccc} \{0\} \otimes Z \otimes X & \xrightarrow{h'} & X' \\ \cap & & \\ I \otimes Z \otimes X & & \downarrow{\scriptstyle p'} \\ \downarrow & & \\ I \otimes Z \otimes B & \xrightarrow{H'} & B'. \end{array}$$

Since $id \otimes p : Z \otimes X \to Z \otimes B$ becomes naturally a numerable principal $G$-bundle, it follows from the covering homotopy theorem for bundle maps (see [2, 7.8]) that there exists a $G$-map $H'' : I \otimes Z \otimes X \to X'$ such that $H'' = h'$ on $\{0\} \otimes Z \otimes X$ and $p'H'' = H'(id \otimes id \otimes p)$. Taking adjoint again, we have $H''' : I \otimes Z \to Map^G(X, X')$ such that $H''' = h$ on $\{0\} \otimes Z$ and $\Phi H''' = H$. Thus $\Phi$ has the $CHP$ for $Z$. The rest follows from the following easy fact: If $p : E \to B$ has the $CHP$ for a $k$-space $Z$, then so does $k(p) : k(E) \to k(B)$. □

The next is a generalization of [7, 2.1]:

**Proposition 4.2.** *Without the assumption that $(G, e)$ is an $NDR$-pair, we have*

1. *If $G$ is compact and $X$ is locally compact, then $\Phi : map^G(X, X') \to map(B, B')$ has the $CHP$ for all locally compact weak Hausdorff spaces and $CW$-complexes.*
2. *If $B$ is locally compact, then $\Phi : map^G(X, X') \to map(B, B')$ has the $CHP$ for all spaces $Z$ with $Z \times X \in WHK$.*

(3) *If $X$ is locally compact, then $\Phi : map^G(X, X') \to map(B, B')$ has the $CHP$ for all spaces in $WHK$. If in addition $X'$ is also locally compact, then $\Phi$ has the $CHP$ for all spaces in $Top$.*
(4) *If $B$ is a $CW$-complex, then $\Phi : map^G(X, X') \to map(B, B')$ has the $CHP$ for locally compact weak Hausdorff spaces and $CW$-complexes.*

*Proof.* The continuity of $\Phi$ is proved in Proposition 3.6.

Assume $G$ is compact and let $Z$ be a locally compact weak Hausdorff space. Then $Z \times X, I \times Z \times X$ and $I \times Z \times B$ are weak Hausdorff $k$-spaces by [14, 4.3]. Assume the commutative diagram

$$\begin{array}{ccccc} \{0\} \times Z & \xrightarrow{h} & map^G(X, X') & \subset & map(X, X') \\ \cap & & \downarrow{\scriptstyle \Phi} & & \\ I \times Z & \xrightarrow{H} & map(B, B'). & & \end{array}$$

Taking adjoints, then by the covering homotopy theorem for bundle maps (see [2, 7.8]), we have a $G$-map $H''$ making the following diagram commutative

$$\begin{array}{ccc} \{0\} \times Z \times X & \xrightarrow{h'} & X' \\ \cap & & \downarrow{\scriptstyle =} \\ I \times Z \times X & \xrightarrow{H''} & X' \\ \downarrow & & \downarrow \\ I \times Z \times B & \xrightarrow{H'} & B'. \end{array}$$

Taking adjoint again, we have the desired covering $H'''$ of $H$, so $\Phi$ has the $CHP$ for locally compact weak Hausdorff spaces. In particular $\Phi$ has the $CHP$ for finite $CW$ complexes and hence for $CW$ complexes. This proves (1).

Assume $B$ is locally compact and $Z \times X \in WHK$. The above proof implies (2).

Note that $X, X', B$ and $B'$ are in $WHK$. Assume $X$ is locally compact. Then $B$ is locally compact, so $\Phi$ has the $CHP$ for spaces in $WHK$, since $p'$ has the $CHP$ for $k$-spaces, by slightly modified theorems of [2, 4.4, 4.8]. Suppose that X' is also locally compact. If $p'$ has the $CHP$ for all spaces in $Top$, then the second part of (3) follows. If $\tau : B' \to I$ is continuous, then $\tau^{-1}(0, 1]$ is locally compact weak Hausdorff and hence $\tau^{-1}(0, 1] \otimes G = \tau^{-1}(0, 1] \times G$. It follows from [2, 4.4, 4.8] that $p'$ has the $CHP$ for all spaces in $Top$.

If $B$ is a $CW$-complex and $Z$ is locally compact weak Hausdorff, then $Z \times X$ and

$Z \times B$ are in $WHK$, so $\Phi$ in (4) has the $CHP$ for $Z$. Thus $\phi$ has also the $CHP$ for $CW$-complexes by the proof of (1). $\square$

Let $(E, p, B)$ be a numerable principal $G$-bundle with a classifying map $f : B \to BG$.

**Definition 4.3.** *We set*

$$\begin{aligned} aut^G(E) &= \{u \in map^G(E,E) \mid uv \simeq_G vu \simeq_G id \ for\ some\ v \in map^G(E,E)\}, \\ aut(B) &= \{u \in map(B,B) \mid uv \simeq vu \simeq id \ forsome\ v \in map(B,B)\}, \\ aut(B;f) &= \{u \in aut(B) \mid fu \simeq f\}, \\ Aut^G(E) &= k(aut^G(E)), \\ Aut(B) &= k(aut(B)), \\ Aut(B;f) &= k(aut(B;f)). \end{aligned}$$

We have the following easily (see [12, p.906]):

**Lemma 4.4.** *$\Phi^{-1}(aut(B;f)) = aut^G(E)$, so $\Phi : Aut^G(E) \to Aut(B;f)$ is a surjective Hurewicz fibration in $WHK$.* $\square$

**Lemma 4.5.** *For any numerable principal $G$-bundle $E$ over $B$, we can choose a universal principal $G$-bundle $(EG, p, BG)$ and a $G$-bundle map*

$$\begin{array}{ccc} E & \xrightarrow{f'} & EG \\ \downarrow & & \downarrow p \\ B & \xrightarrow{f} & BG, \end{array}$$

*where $f$ and $f'$ are embeddings onto closed sets.*

*Proof.* Let $(E', p', B')$ be any universal principal $G$-bundle and let

$$\begin{array}{ccc} E & \xrightarrow{h'} & E' \\ \downarrow & & \downarrow p' \\ B & \xrightarrow{h} & B' \end{array}$$

be a $G$-bundle map. Let $M = B \times I + B'/(b,0) \sim h(b)$ be the mapping cylinder of $h$. Then $M$ is a weak Hausdorff $k$-space by 1.1. Let $q : M' = B \times I + B' \to M$ be the quotient map. Let $i' : B' \to M'$ and $j' : M' \to B'$ be maps defined by $i'(y) = y$, $j'(b,t) = h(b)$, and $j'(y) = y$. Put $i = qi' : B' \to M$. Then $i$ is a homeomorphism onto a closed set. Since $j'(b,0) = j'(h(b))$, we have a map $j : M \to B'$ such that $jq = j'$. We have

$$(\#) \qquad ji = id_B, \text{ and } ij \simeq id_M.$$

Indeed, a homotopy $F : I \times M \to M$ of $ij$ to $id_M$ is made from $F' : I \times M' \to M'$, where $F'(s,y) = y$ and $F'(s,b,t) = (b,st)$.

Let $u : B \to M$ be defined by $u(b) = (b,1)$. Then $ju = h$ so that $u^*j^*E' \cong h^*E' \cong E$. We then have a commutative diagram

$$\begin{array}{ccccc} E \overset{\alpha}{\cong} u^*j^*E' & \xrightarrow{\beta} & j^*E' & \longrightarrow & E' \\ \downarrow & & \downarrow p_1 & & \downarrow \\ B & \xrightarrow{u} & M & \xrightarrow{j} & B'. \end{array}$$

Since $u$ is a homeomorphism onto a closed set, so is $\beta$. By (#), $(j^*E', p_1, M)$ is a universal principal $G$-bundle, thus taking $(EG, p, BG) = (j^*E', p_1, M)$, $f = u$ and $f' = \beta\alpha$, the result follows. □

**Proposition 4.6.** *Let $E$ be a numerable principal $G$-bundle over $B$. Then*

$$[X, Map^G(E, EG)] = 0$$

*for all $X \in WHK$, so $Map^G(E, EG)$ is contractible in the weak sense. If $(X, *)$ is an $NDR$-pair in $WHK$, then*

$$[X, *; Map^G(E, EG), h] = 0$$

*for each $h \in Map^G(E, EG)$.*

*Proof.* Let $X \in WHK$ and consider it as a trivial $G$-space. Let $Y, Z$ be $G$-spaces. Then we have a commutative square

$$\begin{array}{ccc} map(X \otimes Y, Z) & \supset & map^G(X \otimes Y, Z) \\ \downarrow & & \downarrow \\ map(X, Map(Y, Z)) & \supset & map(X, Map^G(Y, Z)) \end{array}$$

and vertical equivalences preserve homotopies, so we have

$$[X \otimes Y, Z]^G \cong [X, Map^G(Y, Z)]$$

and in particular

$$[X \otimes E, EG]^G \cong [X, Map^G(E, EG)].$$

By [2, 7.7] we have $[X \otimes E, EG]^G = 0$. Thus $[X, Map^G(E, EG)] = 0$. Taking $X = \{*\}$, $MapG(E, EG)$ is path-connected. Taking $X = Map^G(E, EG)$, $Map^G(E, EG)$ is contractible in the weak sense. If $(X, *)$ is an $NDR$-pair, then $[X, *; Map^G(E, EG), f] = 0$. □

*Remark.* We can prove an analogous theorem to 4.6 in the theory of Hurewicz fibrations by [3],[4]. In this case base spaces must be $CW$-complexes (see [16, 2.1]).

**Proposition 4.7.** *Let $E$ be a numerable principal $G$-bundle over $B$, and let $X \in WHK$ with a trivial $G$-action. Suppose $X \times E \in WHK$. Then $[X, map^G(E, EG)] = 0$. In particular $map^G(E, EG)$ is essentially contractible in the sense of* [5].

*Proof.* Under the assumptions, we have a commutative square

$$\begin{array}{ccc} map(X \times E, EG) & \supset & map^G(X \times E, EG) \\ \downarrow & & \downarrow \\ map(X, map(E, EG)) & \supset & map(X, map^G(E, EG)). \end{array}$$

Since vertical equivalences preserve homotopies,

$$[X \times E, EG]^G \cong [X, map^G(E, EG)].$$

By [2, 7.7], $[X \times E, EG]^G = 0$. Thus $[X, map^G(E, EG)] = 0$. □

**Lemma 4.8.** *Under the condition of 4.5, we have the following commutative diagram*

$$\begin{array}{ccccc} I^G(E) & = & I^G(E) & = & I^G(E) \\ \cap & & \cap & & \cap \\ Aut^G(E) & \to & Map^G(E, E) & \subset & Map^G(E, EG) \\ \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\ Aut(B; f) & \to & Map(B, B; f, f) & \subset & Map(B, BG; f, id), \end{array}$$

*where $I^G(E) = k(\Phi^{-1}(id_B))$ and three vertical sequences are surjective fibrations in $WHK$.*

*Proof.* Easy. □

**Lemma 4.9.** *Let $X$, $Y$ be topological monoids in $Top$ or $WHK$. Let $p : X \to Y$ be a continuous unital homomorphism having the $CHEP$ for $(Z, *)$ in $Top_*$ or $WHK_*$. Then $F = p^{-1}(1)$ is a submonoid of $X$ and the connecting function*

$$\Delta : [\Sigma Z, *; Y, 1] \to [Z, *; F, 1]$$

*is a homomorphism of semi-groups, where 1 denotes the unit of monoid.*

*Proof.* By definition $\Delta$ is the composition of

$$[\Sigma Z, *; Y, 1] \xrightarrow{p_*^{-1}} [CZ, Z, *; X, F, 1] \to [Z, *; F, 1],$$

where the last function maps $[w]$ into $[w \mid F]$. For any map $u : (\Sigma Z, *) \to (Y, 1)$, we denote by $u' : (CZ, Z, *) \to (X, F, 1)$ a map such that $pu' = uq$, where $q : CZ \to \Sigma Z$ is the quotient map. Hence $p_*([u']) = [u]$. Such a map $u'$ exists by the $CHEP$. As is well known, $[u][v] \in [\Sigma Z, *; Y, 1]$ is represented by $uv : \Sigma Z \to Y$ which is defined by $(uv)(t) = u(t)v(t)$. Since $p$ is a homomorphism, we can take $(uv)' = u'v'$, hence $\Delta([u][v]) = [(u'v') \mid Z] = [u' \mid Z \circ v' \mid Z] = [u' \mid Z][v' \mid Z] = \Delta([u])\Delta([v])$. □

By this lemma and (4.8) we have

**Proposition 4.10.** *If $E$ is a numerable $G$-bundle over $B$ and if $Z \in WHK_*$, then the connecting function*

$$\Delta : [\Sigma Z, *; Map(B, BG), f] \to [Z, *; I^G(E), f']$$

*of the fibration $I^G(E) \to Map^G(E, EG) \to Map(B, BG)$ is a homomorphism of semi-groups for every $Z \in WHK_*$. (Note that $[\Sigma Z, *; Map(B, BG), f] = [\Sigma Z, *; Map(B, BG; f, id), f]$ and $f' = id_E$ if we regard $I^G(E) \subset Map^G(E, E)$.)*
□

**Lemma 4.11.** *Suppose that the morphism in $WHK$, $p : X \to Y$, has the $CHEP$ for $(Z, *) \in WHK_*, y_0 \in Y, F = p^{-1}(y_0)$ is a topological monoid in $WHK$, and that $\cdot : X \otimes F \to X$ is a map such that*

1. *if $x_1, x_2 \in F$, then $x_1 \cdot x_2$ is the product in $F$,*
2. *$(x \cdot x_1) \cdot x_2 = x \cdot (x_1 \cdot x_2), x \in X, x_1, x_2 \in F$, and*
3. *$p(x \cdot x_1) = p(x)$.*

*Then the connecting function $\Delta : [\Sigma Z, *; Y, y_0] \to [Z, *; F, 1]$ of the fibration is a homomorphism of semi-groups.*

*Proof.* Let $u, v : (\Sigma Z, *) \to (Y, y_0)$ be maps. Recall that a map $u + v : \Sigma Z \to Y$ defined by

$$(u+v)(z,t) = \begin{cases} u(z, 2t), & \text{for } 0 \le t \le 1/2 \\ v(z, 2t-1), & \text{for } 1/2 \le t \le 1. \end{cases}$$

Let $u', v' : (CZ, Z, *) \to (X, F, 1)$ be maps such that $pu' = uq$ and $pv' = vq$ where $q : CZ \to \Sigma Z$ is the quotient map. These maps exist by the $CHEP$. Then, by definition, $\Delta[u] = [u' \mid Z]$ and $\Delta[v] = [v' \mid Z]$. We define a map $r : (CZ, Z, *) \to (X, F, 1)$ by

$$r(z,t) = \begin{cases} u'(z, 2t) \cdot v'(z, 0), & \text{for } 0 \le r \le 1/2 \\ v'(z, 2t-1), & \text{for } 1/2 \le t \le 1. \end{cases}$$

This is well defined by (1), and $pr = (u+v)q$, so $\Delta([u][v]) = \Delta[u+v] = [r \mid Z] = [u' \mid Z \cdot v' \mid Z] = [u' \mid Z][v' \mid Z] = \Delta[u]\Delta[v]$. This completes the proof. □

Since $\cdot : Map^G(E, EG) \otimes I^G(E) \to Map^G(E, EG)$, defined by $\alpha \cdot \beta = \alpha\beta$, satisfies the assumptions of lemma 4.11, we obtain the following by using (4.6), (4.10).

**Theorem 4.12.** *If $E$ is a numerable principal $G$-bundle over $B$, then for any $Z \in WHK_*$, we have the following exact sequence of groups and homomorphisms*

$$\begin{aligned} \cdots \longrightarrow [\Sigma^n Z, *; Aut^G(E), id_E] &\xrightarrow{\Phi_*} [\Sigma^n Z, *; Aut(B), id_B] \xrightarrow{f_*} \\ [\Sigma^n Z, *; Map(B, BG), f] &\xrightarrow{\Delta} [\Sigma^{n-1} Z, *; Aut^G(E), id_E] \longrightarrow \\ \cdots \longrightarrow [\Sigma Z, *; Map(B, BG), f] &\longrightarrow [Z, *; Aut^G(E), id_E] \\ &\longrightarrow [Z, *; Aut(B; f), id_B] \longrightarrow 1. \end{aligned}$$

□

## §5. Principal bundles over a suspension space

By Theorem 4.12, we can generalize [12, 2.2] under the assumption that $B \in WHK_*$ and $(G, e)$ is an $NDR$-pair.

**Theorem 5.1.** *If $B, Z \in WHK_*$ and $E$ is a numerable principal $G$-bundle over $\Sigma B$ with the classifying map $f : \Sigma B \to BG$, then we have the following commutative* *diagram with exact rows*

$$\begin{array}{ccc}
0 & \longrightarrow & [\Sigma^{n+1}Z\wedge B,*;\Sigma B,*]/[[\Sigma^{n+1}Z,*;\Sigma B,*],1] \longrightarrow \\
 & & \downarrow \chi \\
0 & \longrightarrow & [\Sigma^{n}Z\wedge B,*;G,e]/\langle[\Sigma^{n}Z,*;G,e],f'\rangle \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
[\Sigma^n Z,*;Map(\Sigma B,\Sigma B),1] & \xrightarrow{ev_*} & [\Sigma^n Z,*;\Sigma B,*] & \longrightarrow & \\
\downarrow f_* & & \downarrow f_* & & \\
[\Sigma^n Z,*;Map(\Sigma B,BG),f] & \xrightarrow{ev_*} & [\Sigma^n Z,*;BG,*] & \longrightarrow & \\
 & & & & [\Sigma^n Z\wedge B,*;\Sigma B,*] \\
 & & & & \downarrow \chi \\
 & & & & [\Sigma^{n-1} Z\wedge B,*;G,e],
\end{array}$$

*where 1 denotes the identity map of* $\Sigma B, f' : (B,*) \to (G,e)$ *is the composition of the adjoint of* $f$ *and a homotopy inverse of the canonical* $H$*-equivalence* $G \to k\Omega BG = Map_0(S_1, BG)$ *given by* [11], $\chi : [\Sigma^m Z\wedge B,*;\Sigma B,*] \to [\Sigma^m Z\wedge B,*;BG,*] \cong [\Sigma^{m-1}Z\wedge B,*;G,e]$ *is the characteristic homomorphism,* $* = f(*) \in BG$*, and* $\langle\ ,f'\rangle$ *is defined by the commutative square*

$$\begin{array}{ccc}
[\Sigma^m Z,*;BG,*] & \xrightarrow{[\ ,f]} & [\Sigma^m Z\wedge B,*;BG,*] \\
\cong\downarrow & & \downarrow\cong \\
[\Sigma^{m-1} Z,*;G,e] & \xrightarrow{\langle\ ,f'\rangle} & [\Sigma^{m-1} Z\wedge B,*;G,e],
\end{array}$$

*where [ , ] denotes the generalized Whitehead product (see* [9]*).*

*Proof.* By [8, 6.44, 2.l00], the evaluation maps for $D = \Sigma B$ or $BG$

$$Map(\Sigma B, D) \to D,$$
$$map(\Sigma B, D) \to D$$

have the $CHEP$ for $NDR$-pairs in $WHK$, so we have a commutative diagram of fibrations

$$\begin{array}{ccccc}
Map_0(\Sigma B,\Sigma B) & \subset & Map(\Sigma B,\Sigma B) & \longrightarrow & \Sigma B \\
\downarrow f_* & & \downarrow f_* & & \downarrow f \\
Map_0(\Sigma B,BG) & \subset & Map(\Sigma B,BG) & \longrightarrow & BG.
\end{array}$$

Taking the homotopy exact sequences of the fibrations, we have the result by [9]. □

## §6. $\mathcal{F}$-FIBRATION MAP THEORY

We shall generalize Proposition 4.1 to $\mathcal{F}$-fibrations of J. P. May [10]. Let $\mathcal{F}$ be a subcategory of $Top$(respectively $WHK$).

**Definition 6.1.** *An $\mathcal{F}$-space is a map $p : E \to B$ in Top (resp. $WHK$) such that $p^{-1}(b)$ is empty or $p^{-1}(b) \in \mathcal{F}$ for each $b \in B$. Notice that if $\mathcal{F} \subset WHK$ then $k(p^{-1}(b)) = p^{-1}(b)$, since $B$ is a $T_1$-space. Let $p : E \to B$ and $q : D \to A$ be $\mathcal{F}$-spaces. An $\mathcal{F}$-map $(g, f) : q \to p$ is a commutative diagram*

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

*in Top (resp. $WHK$) such that $g : q^{-1}(a) \to p^{-1}(f(a))$ is in $\mathcal{F}$ for each $a \in A$.*

*If $A = B$ and $f$ is the identity map, then $g$ is said to be an $\mathcal{F}$-map over $B$. An $\mathcal{F}$-homotopy is an $\mathcal{F}$-map $(H, h)$ of the form*

$$\begin{array}{ccc} I \times D & \xrightarrow{H} & E \\ {\scriptstyle id\times q}\downarrow & & \downarrow{\scriptstyle p} \\ I \times A & \xrightarrow{h} & B. \end{array}$$

*If $A = B$ and $h(s, b) = b$, then $H$ is said to be an $\mathcal{F}$-homotopy over $B$. An $\mathcal{F}$-map $g : D \to E$ over $B$ is an $\mathcal{F}$-homotopy equivalence if there is an $\mathcal{F}$-map $g' : E \to D$ over $B$ such that $g'g$ and $gg'$ are $\mathcal{F}$-homotopic over $B$ to the respective identity maps. An $\mathcal{F}$-space $p : E \to B$ is said to be $\mathcal{F}$-homotopy trivial if it is $\mathcal{F}$-homotopy equivalent to the projection $\pi_1 : B \times F(resp. B \otimes F) \to B$ for some $F \in \mathcal{F}$.*

*Remark.* We can work with the assumption of the surjectivity of $\mathcal{F}$-spaces.

**Definition 6.2.** *An $\mathcal{F}$-space $p : E \to B$ has the $\mathcal{F}$-covering homotopy property (abbreviated $\mathcal{F}$-CHP) for an $\mathcal{F}$-space $q : D \to A$ if for every $\mathcal{F}$-map $(g, f) : q \to p$ and every homotopy $h : I \times A \to B$ of $f$, there exists a homotopy $H : I \times D \to E$ of $g$ such that the pair $(H, h)$ is an $\mathcal{F}$-homotopy. An $\mathcal{F}$-space is an $\mathcal{F}$-fibration if it has the $\mathcal{F}$-CHP for every $\mathcal{F}$-space.*

For $\mathcal{F}$-spaces $p : E \to B$ and $q : D \to A$, we use the following notations:

$$\begin{aligned} map\mathcal{F}(q, p) &= the\ set\ of\ \mathcal{F}\text{-}maps\ from\ q\ to\ p \\ &\subset map(D, E) \times map(A, B), \\ \Phi = \Phi_{q,p} &: map\mathcal{F}(q, p) \to map(A, B), \Phi(g, f) = f, \\ \phi = \phi_{q,p} &: map\mathcal{F}(q, p) \to map(D, E), \phi(g, f) = g, \\ Map\mathcal{F}(q, p) &= k(map\mathcal{F}(q, p)). \end{aligned}$$

Note that, if $q$ is surjective, then $\phi : map\mathcal{F}(q,p) \to map(D,E)$ is injective.

**Proposition 6.3.** *Let $\mathcal{F} \subset WHK$. With the above notations, we have*

1. $\Phi : Map\mathcal{F}(q,p) \to Map(A,B)$ *is continuous.*
2. *If $p$ is an $\mathcal{F}$-fibration, then $\Phi : Map\mathcal{F}(q,p) \to Map(A,B)$ has the $CHP$ in $WHK$.*
3. *If $p$ has the $\mathcal{F}-CHP$ for $id \otimes q : K \otimes D \to K \otimes A (K \in WHK)$, then $\Phi : Map\mathcal{F}(q,p) \to Map(A,B)$ has the $CHP$ for $K$.*
4. *If $D, E \in WH$ and $q$ is surjective and proclusive, then*

$$map(D,E) \supset Im(\phi) \xrightarrow{\Phi\phi^{-1}} map(A,B)$$

*is continuous on each compact set. Hence*

$$\Phi\phi^{-1} : k(Im(\phi)) \to Map(A,B)$$

*is continuous, and*

$$\phi : Map\mathcal{F}(q,p) \cong k(Im(\phi))$$

*is a homeomorphism.*

*Proof.* Since the projection is continuous, (1) follows. Under the notations of (3), suppose the commutative diagram

$$\begin{array}{ccc} \{0\} \times K & \xrightarrow{h} & Map\mathcal{F}(q,p) \\ \cap & & \downarrow \Phi \\ I \times K & \xrightarrow{H} & Map(A,B). \end{array}$$

The adjoint of the composition of $h$ with the inclusion and the projection

$$Map\mathcal{F}(q,p) \to Map(D,E) \otimes Map(A,B) \to Map(D,E)$$

is the map $h' : \{0\} \otimes K \otimes D \to E$. By the assumption, there exists a map $J$ such that $(J, ad(H))$ is an $\mathcal{F}$-map and the following diagram is commutative

$$\begin{array}{ccc} \{0\} \otimes K \otimes D & \xrightarrow{h'} & E \\ \cap & & \downarrow = \\ I \otimes K \otimes D & \xrightarrow{J} & E \\ {\scriptstyle id\otimes id\otimes q}\downarrow & & \downarrow p \\ I \otimes K \otimes A & \xrightarrow{ad(H)} & B. \end{array}$$

The adjoint of $J$ and the map $H$ give a map $I \otimes K \to Map\mathcal{F}(q,p)$ extending $h$ and lifting $\Phi$. This proves (3).

(2) can be proved by the same way as (3).

To prove (4), suppose that $D, E \in WH$ and $q$ is surjective and proclusive. The surjectivity of $q$ implies that $\phi : map\mathcal{F}(q,p) \to Im(\phi)$ is a bijection. Consider the functions

$$map(D,E) \supset Im(\phi) \xrightarrow{\phi^{-1}} map\mathcal{F}(q,p) \xrightarrow{\Phi} map(A,B).$$

Let $T \subset Im(\phi) \subset map(D,E)$ be a compact set, $g \in T$, and suppose that $\Phi\phi^{-1}(g) \in W(K,U)$, where $U \subset B$ is open and $K \subset A$ is compact.

Since $D, E \in WH$, $ev : Map(D,E) \otimes D \to E$ is continuous by [l5,3.7]. Thus $ev : T \otimes D \to E$ is continuous. Since $T$ is compact and $q$ is proclusive, $id \times q : T \otimes D \to T \otimes A$ is proclusive by [14, 4.5], [11, 2.2] and [15, 3.8]. Hence we have a map $\Phi'$ making the following diagram commutative

$$\begin{array}{ccc} T \otimes D & \xrightarrow{ev} & E \\ {\scriptstyle id\times q}\downarrow & & \downarrow{\scriptstyle p} \\ T \otimes A & \xrightarrow{\Phi'} & B \end{array}$$

It follows that there exist neighborhoods $V$ of $g$ in $T$ and $V'$ of $K$ in $A$ such that $\Phi'(V, V') \subset U$. In particular, if $v \in V$, then $\Phi\phi^{-1}(v)(K) = \Phi'(v, K) \subset U$, so $\Phi\phi^{-1}(V) \subset W(K,U)$, thus $\Phi\phi^{-1}$ is continuous on $T$. $\square$

**Example 6.4.** Let $G$ be a topological group in $WHK$ and $\mathcal{F}$ denote the category whose objects are right $G$-spaces $Y$ such that, for all $y \in Y$, the function $y' : G \to Y$ defined by $y'(g) = yg$ is a homeomorphism and whose morphisms of $\mathcal{F}$ are $G$-maps. A principal $G$-bundle is a surjective $\mathcal{F}$-space by 2.4. If $q : D \to A$ and $p : E \to B$ are principal $G$-bundles, then the maps map$\mathcal{F}(q,p) \subset map(D,E) \times map(A,B) \to map(D,E)$ give the homeomorphism

$$Map\mathcal{F}(q,p) \to Map^G(D,E)$$

by 6.3 (4). Since a numerable principal $G$-bundle has the $\mathcal{F}$-$CHP$ for numerable principal $G$-bundles by [2, 7.8], 4.1 is a special case of 6.3 (3).

## §7. Fibration map theory

Let $\mathcal{F}$ be the category whose objects are weak Hausdorff $k$-spaces and morphisms are homotopy equivalences. In this case the $\mathcal{F}$-fibration map theory is called the fibration map theory in [16](cf. [3], [4]). He assumed base spaces of fibrations to be $CW$-complexes. We shall see that this assumption will not be necessary for some results of [16].

If $p : E \to B$ and $f : A \to B$ are maps in $WHK$, then the pull back of them is denoted by $f^*E$, that is,

$$f^*E = k\{(a, x) \in A \times E \mid f(a) = p(x)\}.$$

**Lemma 7.1.** *If $p : E \to B$ is a fibration in $WHK$, then $p$ is an $\mathcal{F}$-fibration.*

*Proof.* Let $q : D \to A$ be any $\mathcal{F}$-space, $(g, f) : q \to p$ an $\mathcal{F}$-map, and let $h : I \times A \to B$ be a homotopy of $f$. By the $CHP$, there exists a map $H : I \times D \to E$ such that $pH = h(id \times q)$ and $H_0 = g$. We shall prove that $(H, h)$ is an $\mathcal{F}$-homotopy, that is, $H_t : q^{-1}(a) \to p^{-1}(h_t(a))$ is a homotopy equivalence for every $a \in A$ and every $t \in I$, by showing that the map $L : I \times q^{-1}(a) \to h^*E$ making the following diagram commutative is a homotopy equivalence on each fibre

$$\begin{array}{ccccc} I \times q^{-1}(a) & \xrightarrow{L} & h^*E & \longrightarrow & E \\ \Big\downarrow{\scriptstyle id\times q} & & \Big\downarrow{\scriptstyle p'} & & \Big\downarrow{\scriptstyle p} \\ I \times \{a\} & \xrightarrow{=} & I \times \{a\} & \xrightarrow{h} & B. \end{array}$$

Since $I \times \{a\}$ is contractible, it follows by [8, 6.57] that there exists a fibre homotopy equivalence $\alpha$, which makes the following diagram commutative

$$\begin{array}{ccc} I \times (h^*E)_{(0,a)} & \xrightarrow{\alpha} & h^*E \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p'} \\ I \times \{a\} & \xrightarrow{=} & I \times \{a\}. \end{array}$$

Let $\beta$ be a fibre homotopy inverse of $\alpha$, and let $\theta : I \times q^{-1}(a) \to h^*E \to I \times (h^*E)_{(0,a)} \to (h^*E)_{(0,a)}$ be the composition of $L$, $\beta$ and the projection. By the assumption, $\theta_0$ is a homotopy equivalence, hence so are $\theta_t = \beta_t L_t$ and $\alpha_t\beta_t L_t$ for every $t$. Since the latter is homotopic to $L_t$, $L_t$ is a homotopy equivalence. This completes the proof. □

By 6.3(2) and 7.1, we have

**Corollary 7.2.** *If $p : E \to B$ is a fibration in $WHK$ and $q : D \to A$ is a map in $WHK$, then*

$$\Phi : Map\mathcal{F}(q, p) \to Map(A, B)$$

*is a fibration in $WHK$.* □

Under the notations of 7.2, in addition, if $q$ is also a surjective and proclusive fibration in $WHK$, for example by [8, 6.36], or if $q$ is a surjective fibration in $Top$ such that $D, A \in WHK$ and $A$ is locally path-connected, then $Map\mathcal{F}(q, p) \cong k(Im(\phi)) = G^*(D, E)$ by 6.3(4), where $G^*$ is the notation of [16].

**Theorem 7.3.** *Let $q : D \to A$ and $p : E \to B$ be fibrations in $WHK$ and $(A, A_0)$ be an $NDR$-pair in $WHK$. Set $D_0 = q^{-1}(A_0), q_0 = q \mid D_0 : D_0 \to A_0$, and let $i : D_0 \to D$ be the inclusion. Then*

$$i^* : Map\mathcal{F}(q, p) \to Map\mathcal{F}(q_0, p)$$

*is a fibration in $WHK$.*

*Proof.* Suppose a commutative diagram

$$\begin{array}{ccc} \{0\} \times K & \xrightarrow{h} & Map\mathcal{F}(q, p) \\ \cap & & \downarrow i^* \\ I \times K & \xrightarrow{H} & Map\mathcal{F}(q_0, p). \end{array}$$

Taking the adjoints, we have the commutative diagram

$$\begin{array}{ccccc} I \otimes K \otimes D & \supset & \{0\} \otimes K \otimes D \cup I \otimes K \otimes D_0 & \xrightarrow{(\phi h)' \cup (\phi H)'} & E \\ \downarrow & & \downarrow & & \downarrow p \\ I \otimes K \otimes A & \supset & \{0\} \otimes K \otimes A \cup I \otimes K \otimes A_0 & \xrightarrow{(\Phi h)' \cup (\Phi H)'} & B, \end{array}$$

where the prime $'$ denotes the adjoint. Since $(A, A_0)$ is an $NDR$- pair, it follows that $(I, 0) \otimes K \otimes (A, A_0)$ is also an $NDR$-pair, hence $(\Phi h)' \cup (\Phi H)'$ can be extended to a map $f : I \otimes K \otimes A \to B$, and we have the following commutative diagram

$$\begin{array}{ccc} \{0\} \otimes K \otimes D \cup I \otimes K \otimes D_0 & \xrightarrow{(\phi h)' \cup (\phi H)'} & E \\ \cap & & \\ I \otimes K \otimes D & & \downarrow p \\ id \otimes q \downarrow & & \\ I \otimes K \otimes A & \xrightarrow{f} & B. \end{array}$$

By [8, 6.42], $(D, D_0)$ is an $NDR$-pair, and so is $(I, 0) \otimes K \otimes (D, D_0)$. Hence there exists a map $g : I \otimes K \otimes D \to E$ extending $(\phi h)' \cup (\phi H)'$ and lifting $f(id \otimes q)$. By the same proof of 7.1, we know that $(g, f)$ is an $\mathcal{F}$-homotopy. Then $g' : I \otimes K \to Map\mathcal{F}(q, p)$, the adjoint of $g$, is an extension of $h$ and $a$ lifting of $H$. This completes the proof. □

Let $q : D \to A$ and $p : E \to B$ be fibrations in $WHK$, $A_0$ a $k$-subspace of $A$, $D_0 = q^{-1}(A_0)$ the $k$-subspace of $D$, and let $q_0 : D_0 \to A_0$ be the restriction of $q$. Let $(g, f) : q_0 \to p$ be an $\mathcal{F}$-map. Under this situation we define

**Definition 7.4.** *We set*

$$\begin{aligned} Map\mathcal{F}_{(g,f)}(q\ mod\ q_0, p) &= k\{(u,v) \in Map\mathcal{F}(q,p); u \mid D_0 = g, v \mid A_0 = f\} \\ &= \{(u,v) \in Map\mathcal{F}(q,p); u \mid D_0 = g, v \mid A_0 = f\}. \end{aligned}$$

**Theorem 7.5.** *If $(A, A_0)$ is an NDR-pair in $WHK$, then, under the above situation, the restriction of $\Phi$:*

$$Map\mathcal{F}_{(g,f)}(q\ mod\ q_0, p) \to Map_f(A\ mod\ A_0, B)$$

*is a fibration in $WHK$, where $Map_f(A\ mod\ A_0, B) = k\{w \in Map(A,B); w \mid A_0 = f\} = \{w \in Map(A,B); w \mid A_0 = f\}$.*

*Proof.* Let $K \in WHK$ and suppose the left square of

$$\begin{array}{ccccc} \{0\} \otimes K & \xrightarrow{j} & Map\mathcal{F}_{(g,f)}(q\ mod\ q_0, p) & \xrightarrow{i} & Map\mathcal{F}(q,p) \\ \cap & & \downarrow \Phi & & \downarrow \Phi \\ I \otimes K & \xrightarrow{h} & Map_f(A\ mod\ A_0, B) & \xrightarrow{i} & Map(A,B) \end{array}$$

is commutative, where $i$ are the canonical injections. Then we have the commutative diagram(note that $\phi : Map\mathcal{F}(q,p) \to Map(D,E)$)

$$\begin{array}{ccc} I \otimes K \otimes D_0 \cup \{0\} \otimes K \otimes D & \xrightarrow{m \cup j'} & E \\ \cap & & \\ I \otimes K \otimes D & & \downarrow p \\ \downarrow & & \\ I \otimes K \otimes A & \xrightarrow{h'} & B, \end{array}$$

where $m$ is the composition of the projection $I \otimes K \otimes D_0 \to D_0$ and $g$, and $j'$ and $h'$ are the adjoints of $\phi ij$ and $ih$ respectively. Since $(I,0) \otimes K \otimes (D, D_0)$ is an $NDR$-pair, there exists a map $n : I \otimes K \otimes D \to E$ extending $m \cup j'$ and lifting $h'(id \otimes q)$. By the proof of 6.1, $(n, h')$ is an $\mathcal{F}$-map. As is easily seen, the adjoint of $n$ factorizes into

$$I \otimes K \xrightarrow{n'} Map\mathcal{F}_{(g,f)}(q\ mod\ q_0, p) \to Map\mathcal{F}(q,p) \to Map(D,E),$$

and $n'$ is the desired map. $\square$

**Theorem 7.6.** *Let $q : D \to A$ and $p : E \to B$ be fibrations in $WHK$ such that $A$ is numerably categorical in the sense of* [8] *(for example, when $A$ is a $CW$- complex). Then each non-empty fibre of the fibration $\Phi_{q,p} : Map\mathcal{F}(q,p) \to Map(A,B)$ has the same homotopy type as $\Phi^{-1}_{p,q}(id_A)$.*

*Proof.* Let $f \in Map(A,B)$ be any map such that $\Phi^{-1}_{q,p}(f)$ is non-empty. Let

$$\begin{array}{ccc} f^*E & \xrightarrow{j} & E \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

be the commutative square of the induced fibration. Then we have the following commutative square

$$\begin{array}{ccc} Map\mathcal{F}(q,r) & \xrightarrow{j_* \times f_*} & Map\mathcal{F}(q,p) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ Map(A,A) & \xrightarrow{f_*} & Map(A,B). \end{array}$$

By the last square, we have a map

$$\xi : \Phi^{-1}_{q,r}(id_A) \to \Phi^{-1}_{q,p}(f).$$

Note that the composition of $\Phi^{-1}(f) \subset map\mathcal{F} \subset map(D,E) \times map(A,B) \to map(D, A \times E)$ is continuous and has the image in $map(D, f^*E) \subset map(D, A \times E)$, where the last map assigns $q \times g : D \to A \times E$ *to* $(g,h)$. For simplicity, we denote the image of $(g,f)$ under this map by $g'$. Since $A$ is numerably categorical, $g'$ is a fibre homotopy equivalence by [2, 6.3] (cf.[8, 7.58]). We have a map

$$\Phi^{-1}_{q,p}(f) \to \Phi^{-1}_{q,r}(id_A)$$

assigning to $(g,f)$ the pair $(g', id_A)$. As is easily seen, this is the inverse of $\xi$, so $\xi$ is a homeomorphism.

Choose arbitrary $(g_0, f) \in \Phi^{-1}_{q,p}(f)$. Let $g''_0 : f^*E \to D$ be a fibre homotopy inverse of $g'_0$. We have the commutative diagram

$$\begin{array}{ccccc} Map\mathcal{F}(q,r) & \xrightarrow{g''_{0*} \times id} & Map\mathcal{F}(q,q) & \xrightarrow{g'_{0*} \times id} & Map\mathcal{F}(q,r) \\ \downarrow{\scriptstyle \Phi_{q,r}} & & \downarrow{\scriptstyle \Phi_{q,q}} & & \downarrow{\scriptstyle \Phi_{q,r}} \\ Map(A,A) & = & Map(A,A) & = & Map(A,A). \end{array}$$

Let $H : I \times D \to D$ and $L : I \times f^*E \to f^*E$ be fibre homotopy maps from suitable identity map to $g_0''g_0'$ and $g_0'g_0''$ respectively. We define maps

$$\alpha : \Phi^{-1}_{q,r}(id_A) \to \Phi^{-1}_{q,q}(id_A),$$
$$\beta : \Phi^{-1}_{q,q}(id_A) \to \Phi^{-1}_{q,r}(id_A)$$

by the restrictions of $g_{0*}'' \times id$ and $g_{0*}' \times id$ respectively. Then

$$\beta\alpha(u, id_A) = (g_0'g_0''u, id_A),$$
$$\alpha\beta(v, id_A) = (g_0''g_0'v, id_A).$$

Let $L' : I \times \Phi^{-1}_{q,r}(id_A) \to \Phi^{-1}_{q.r}(id_A)$ and $H' : I \times \Phi^{-1}_{q,q}(id_A) \to \Phi^{-1}_{q,q}(id_A)$ be defined by $L'(t, u, id) = (L_t u, id)$ and $H'(t, v, id_A) = (H_t v, id_A)$. Then $L'$ is a homotopy of id to $\beta\alpha$ and $H'$s a homotopy of $id$ to $\alpha\beta$. This implies that $\alpha$ is a homotopy equivalence of which a homotopy inverse is $\beta$. Hence $\Phi^{-1}_{q,p}(f)$ and $\Phi^{-1}_{q,q}(id_A)$ have the same homotopy type. □

**Theorem 7.7.** *Let $p : E \to B$ be a fibration in $WHK$, $(B, A)$ an NDR-pair, and let $q : D = p^{-1}(A) \to A$ be the restriction of $p$. Then*

$$i^* : \Phi^{-1}_{q,p}(id_B) \to \Phi^{-1}_{q,q}(id_A)$$

*is a fibration in $WHK$, where $i : D \subset E$ is the inclusion.*

*Proof.* Let $K \in WHK$ and suppose a commutative square

$$\begin{array}{ccccc}
\{0\} \otimes K & \xrightarrow{g} & \Phi^{-1}_{p,p}(id_B) & \xrightarrow{\phi} & Map(E, E) \\
\cap & & \downarrow{\scriptstyle i^*} & & \\
I \otimes K & \xrightarrow{f} & \Phi^{-1}_{q,q}(id_A) & \xrightarrow{\phi} & Map(D, D).
\end{array}$$

Then we have the commutative diagram

$$\begin{array}{ccc}
\{0\} \otimes K \otimes E \cup I \otimes K \otimes D & \xrightarrow{g' \cup f'} & E \\
\cap & & \\
I \otimes K \otimes E & & \downarrow{\scriptstyle p} \\
{\scriptstyle id \otimes p}\downarrow & & \\
I \otimes K \otimes B & \xrightarrow{\pi} & B,
\end{array}$$

where $g'$ and $f'$ are adjoints of $\phi g$ and $\phi f$ respectively, and $\pi$ is the projection. By the $CHEP$, there exists a map $h : I \otimes K \otimes E \to E$ extending $g' \cup f'$ and lifting $\pi(id \otimes p)$. By the proof of 7.1, $(h, \pi) \in Map\mathcal{F}(id \circ p, p)$. Note that $I \otimes K \to Map(B, B)$, the adjoint of $\pi$, is constant into $id_B$. Thus the adjoint of $h$ defines the map $I \otimes K \to \Phi_{p,p}^{-1}(id_B)$ which is an extension of $g$ and a lifting of $f$. □

*Remark.* For function spaces, we have considered the compact open topology. We note that we can consider the quasi-compact open topology, and that the $k$-ification of this topology coincides with the $k$-ification of the compact-open topology in $WHK$.

## References


1. N. Bourbaki, *Éléments de mathématique topologie générale*, Chapitres 1 et 2, Hermann, Paris, l965.
2. A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. **78** (1963), 223-255.
3. D. H. Gottlieb, *On fibre spaces and the evaluation map*, Ann. of Math. **87** (1968), 42-55.
4. D. H. Gottlieb, *Correction to "On fibre spaces and the evaluation map"*, Ann. of Math **91** (1970), 640-642.
5. D. H. Gottlieb, *Applications of bundle map theory*, Trans. Amer. Math. Soc. **171** (1972), 23-50.
6. D. Husemoller, *Fibre bundles, 2nd ed*, Springer Verlag, Berlin, 1975.
7. I. M. James, *The space of bundle maps*, Topology **2** (1963), 45-59.
8. I. M. James, *General topology and homotopy theory*, Springer Verlag, Berlin, 1984.
9. G. E. Lang, *The evaluation map and EHP sequences*, Pacific J. Math. **44** (1973), 201-210.
10. J. P. May, *Classifying spaces and fibrations*, vol. 155, Mem. Amer. Math. Soc., 1975.
11. M. C. McCord, *Classifying spaces and infinite symmetric products*, Trans. Amer. Math. Soc. **146** (1969), 273-298.
12. H. Ōshima and K. Tsukiyama, *On the group of equivariant self equivalences of free actions*, Publ. RIMS. Kyoto Univ. **22** (1986), 905-923.
13. R. S. Palais, *The classification of G-spaces*, vol. 36, Mem. Amer. Math. Soc., 1960.
14. N. E. Steenrod, *A convenient category of topological spaces*, Mich. Math. J. (1967), 133-152.
15. R. M. Vogt, *Convenient categories of topological spaces for homotopy theory*, Archiv Math. **22** (1971), 545-555.
16. T. Yamanoshita, *On the spaces of self homotopy equivalences for fibre spaces*, Kodai Math. J. **10** (l987), 127-142.



Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-Ku, Osaka 558-0022, Japan

*E-mail address*: i38599@m-unix.cc.u-tokyo.ac.jp

Department of Mathematics, Faculty of Education, Shimane University, Matsue 690-8504, Japan

*E-mail address*: tukiyama@edu.shimane-u.ac.jp