

CHARACTERIZATION THEOREM OF 4-VALUED DE MORGAN LOGIC

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ABSTRACT. In this paper we give an axiom system of a non-linear 4-valued logic which we call a de Morgan logic (*ML*), whose Lindenbaum algebra is the de Morgan algebra with implication (*MI*-algebra), and show that

- (1) For every *MI*-algebra L , there is a quotient *MI*-algebra $L^\#$ such that it is embeddable to the simplest 4-valued *MI*-algebra $\mathbf{M} = \{0, a, b, 1\}$;
- (2) The Lindenbaum algebra of *ML* is the *MI*-algebra;
- (3) The completeness theorem of *ML* is established;
- (4) *ML* is decidable.

1. INTRODUCTION

It is well known the relation between logics and algebras through the property of Lindenbaum algebras. For example, the Lindenbaum algebra for the classical propositional logic (*CPL*) is a Boolean algebra and that of the intuitionistic propositional logic (*IPL*) is a Heyting one. On the other hand, *CPL* is characterized by the simplest non-trivial Boolean algebra $2 = \{0, 1\}$, that is, a formula A is provable in *CPL* if and only if (iff) $\tau(A) = 1$ for every function $\tau : \Pi \rightarrow 2$. In [3] or [4], we have an axiom system of Kleene logic (*KL*) and show in [3] that *KL* is characterized by the simplest non-trivial Kleene algebra $3 = \{0, 1/2, 1\}$. Now the next question arises:

(Question) What is a logic characterized by 4-valued algebra?

Of course, Rasiowa [5] has already given the axiom system of n -valued logic which is determined by n -valued Post-algebras. But Post-algebras are linear and so that these logics are not determined by 4-valued non-linear de Morgan algebra $\{0, a, b, 1\}$ below. In [1], it is proved that the class of de Morgan algebras with no implication is functional free for the algebra $\{0, a, b, 1\}$. Hence there is a question about the functional freeness of de Morgan algebras with implication.

Moreover, in the relevance logic investigated in [2], the set of first-degree formulas of the logic is determined by 4-valued de Morgan algebra $\{0, a, b, 1\}$, that is, for the zero-degree formulas (i.e., formulas with no implication symbols) A

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and B , $A \rightarrow B$ is provable in the relevance logic iff $\tau(A) \leq \tau(B)$ for every $\tau : \Pi \rightarrow \{0, a, b, 1\}$. But the algebra $\{0, a, b, 1\}$ does not essentially include the implication. Hence, from the view point of relevance logics, it is also interesting to find logics characterized by 4-valued de Morgan algebras with implication.

In this paper, we give an answer to these questions. That is, we shall define de Morgan algebras with implication (*MI*-algebras) and give an axiom system of some logic called de Morgan logic (*ML*), which is characterized by 4-valued de Morgan algebra \mathbf{M} with implication. More precisely, we show that

- (1) For every *MI*-algebra L there exists a quotient *MI*-algebra $L^\#$ such that it is embeddable to the simplest *MI*-algebra $\mathbf{M} = \{0, a, b, 1\}$;
- (2) The Lindenbaum algebra of *ML* is a *MI*-algebra;
- (3) The completeness theorem of *ML* is established;
- (4) *ML* is decidable.

2. DE MORGAN ALGEBRA WITH IMPLICATION

In this section we define a de Morgan algebra with implication. In this paper we call it an *MI*-algebra. By an *MI*-algebra, we mean an algebraic structure $L = (L; \wedge, \vee, \rightarrow, N, 0, 1)$ of type $(2, 2, 2, 1, 0, 0)$ such that

- (1) $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice;
- (2) $N : L \rightarrow L$ is a map satisfying the following conditions
 - (N1) $N0 = 1, N1 = 0$
 - (N2) $x \leq y$ implies $Ny \leq Nx$
 - (N3) $N(Nx) = x$, $N(Nx)$ is denoted by N^2x
- (3) the implication \rightarrow satisfies
 - (I1) $y \leq x \rightarrow y$
 - (I2) $x \leq y$ implies $x \rightarrow y = 1$
 - (I3) $Nx \wedge Ny \leq (x \rightarrow y) \vee y$
 - (I4) $Nx \wedge y \wedge N(x \rightarrow y) = 0$
 - (I5) $x \wedge (x \rightarrow y) \leq y \vee Ny$
 - (I6) $x \wedge (x \rightarrow y) \leq Nx \vee y$
 - (I7) $x \wedge Nx \wedge (x \rightarrow y) \leq N(x \rightarrow y) \vee y$

Example: As a model of *MI*-algebras we list $\mathbf{M} = \{0, a, b, 1\}$, where $0 < a = Na, b = Nb < 1$ and a, b are not comparable:

$$\mathbf{M} = \{0, a, b, 1\}$$

\rightarrow	0	a	b	1
0	1	1	1	1
a	a	1	b	1
b	b	a	1	1
1	0	a	b	1

Let L be any *MI*-algebra. A non-empty subset F of L is called a filter when it satisfies the conditions:

- (f1) $x, y \in F$ imply $x \wedge y \in F$;
- (f2) $x \in F$ and $x \leq y$ imply $y \in F$.

A filter F is called proper when it is a proper subset of L , that is, $0 \notin F$. We define two kinds of filters of L . By a maximal filter M , we mean the proper filter M such that $M \subseteq G$ implies $M = G$ for any proper filter G . A proper filter P is called prime if $x \vee y \in P$ implies $x \in P$ or $y \in P$ for every $x, y \in L$.

It is easy to show the next lemmas, so we omit their proofs.

Lemma 1. *If $x \in L$ and $x \neq 0$, then there is a maximal filter M of L such that $x \in M$.*

Lemma 2. *If M is a maximal filter of L , then it is a prime filter.*

Let M be an arbitrary maximal filter of L . We define the subsets L_j ($j \in \{0, a, b, 1\}$) of L :

$$L_0 = \{x \in L \mid x \notin M, Nx \in M\}$$

$$L_a = \{x \in L \mid x \in M, Nx \in M\}$$

$$L_b = \{x \in L \mid x \notin M, Nx \notin M\}$$

$$L_1 = \{x \in L \mid x \in M, Nx \notin M\}$$

Now we define an equivalence relation \sim as follows: For $x, y \in L$,

$$x \sim y \iff \exists L_j (x, y \in L_j) .$$

Since each subset L_j of L is an equivalence class, we denote it simply by j^\sharp , that is, $L_j = j^\sharp$. The relation \sim is obtained alternatively.

Lemma 3. *$x \sim y$ if and only if $x \in M \Leftrightarrow y \in M$ and $Nx \in M \Leftrightarrow Ny \in M$.*

Lemma 4. *The relation \sim is a congruence on L .*

Proof. It suffices to show that

$$(1) \ x \sim y \text{ and } p \sim q \text{ imply } (x \wedge p) \sim (y \wedge q);$$

$$(2) \ x \sim y \text{ implies } Nx \sim Ny;$$

$$(3) \ x \sim y \text{ and } p \sim q \text{ imply } (x \rightarrow p) \sim (y \rightarrow q);$$

We only consider the case of (3).

Case (3): Suppose that $x \sim y$ and $p \sim q$. It is sufficient to show that

$$(x \rightarrow p) \in M \iff (y \rightarrow q) \in M \text{ and } N(x \rightarrow p) \in M \iff N(y \rightarrow q) \in M.$$

We prove that the left-hand statement implies the right-hand one in the first case. Suppose that $(x \rightarrow p) \in M$. We need to show that $(y \rightarrow q) \in M$. If $p \in M$, then we have $q \in M$ by assumption. Thus it follows from (I1) that $y \rightarrow q \in M$.

We suppose that $p \notin M$ (i.e. $q \notin M$). We have two cases (a) $x \notin M$ or (b) $x \in M$.

• Case (a) $x \notin M$ (i.e. $y \notin M$): From (I4) $y \vee Nq \vee (y \rightarrow q) = 1 \in M$ and assumption $y \notin M$, we have $Nq \in M$ or $(y \rightarrow q) \in M$. It goes well if $(y \rightarrow q) \in M$. If $Nq \in M$ (i.e. $Np \in M$), there are two subcases (a1) $Nx \in M$ or (a2) $Nx \notin M$.

Subcase (a1): If $Nx \in M$ (i.e. $Ny \in M$), then we have $Ny \wedge Nq \leq (y \rightarrow q) \vee q \in M$ by (I3). Since $q \notin M$, it is the case that $y \rightarrow q \in M$.

Subcase (a2): If $Nx \notin M$, then from (I3) we get that $N(x \rightarrow p) \wedge Np \leq x \vee p \notin M$. Since $Np \in M$, it follows that $N(x \rightarrow p) \notin M$. But in this case we obtain that $x \rightarrow p \notin M$ by (I7). This is a contradiction.

• Case (b) $x \in M$: We get from (I5) and (I6) that $Nx \in M$ (i.e. $Ny \in M$) and $Nq \in M$. It follows from (I3) that $Ny \wedge Nq \leq (y \rightarrow q) \vee q \in M$ and hence that $y \rightarrow q \in M$ by $q \notin M$.

Thus $x \rightarrow p \in M$ implies $y \rightarrow q \in M$. The other cases are proved similarly.

Hence the relation \sim is the congruent relation on L .

Since \sim is congruent, we can consistently define operations \wedge , \vee , \rightarrow , and N in $L^\# = \{0^\#, a^\#, b^\#, 1^\#\}$: For every $i^\#, j^\# \in L^\#$,

$$\begin{aligned} i^\# \wedge j^\# &= (i \wedge j)^\# \\ i^\# \vee j^\# &= (i \vee j)^\# \\ i^\# \rightarrow j^\# &= (i \rightarrow j)^\# \\ N(i^\#) &= (Ni)^\# \end{aligned}$$

The general theory of universal algebras shows the next theorem.

Theorem 1. *Let L be a MI-algebra and M be a maximal filter of L . There exists a quotient MI-algebra $L^\# = (L^\#; \wedge, \vee, N, \rightarrow, 0^\#, 1^\#)$ which is embeddable to the MI-algebra \mathbf{M} .*

Proof. The map $\xi : L^\# \rightarrow \mathbf{M}$ defined by $\xi(x^\#) = j$ when $x \in L_j$ gives the desired result.

3. DE MORGAN LOGIC ML

In this section we shall define a de Morgan logic ML . As a language we use a countable set of propositional variables $p_1, p_2, \dots, p_n, \dots$ and logical symbols \wedge, \vee, \neg , and \rightarrow . We denote the set of propositional variables by Π , that is, $\Pi = \{p_1, p_2, \dots, p_n, \dots\}$. The formulas of ML are defined as usual. Let A, B, C, \dots be arbitrary formulas of ML . In the following we list an axiom system of ML .

Axioms;

- (A1) $A \rightarrow A$
- (A2) $A \wedge B \rightarrow A$
- (A2)' $A \rightarrow A \vee B$
- (A3) $A \wedge B \rightarrow B \wedge A$
- (A3)' $A \vee B \rightarrow B \vee A$
- (A4) $A \rightarrow \neg\neg A$
- (A5) $\neg\neg A \rightarrow A$
- (A6) $A \rightarrow (B \rightarrow A)$
- (A7) $\neg A \wedge \neg B \rightarrow (A \rightarrow B) \vee B$
- (A8) $A \vee \neg B \vee (A \rightarrow B)$
- (A9) $A \wedge (A \rightarrow B) \rightarrow B \vee \neg B$
- (A10) $A \wedge (A \rightarrow B) \rightarrow \neg A \vee B$
- (A11) $A \wedge \neg A \wedge (A \rightarrow B) \rightarrow \neg(A \rightarrow B) \vee B$
- (A12) $\neg(A \wedge B) \rightarrow \neg A \vee \neg B$
- (A13) $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$
- (A14) $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
- (A15) $(A \vee B) \wedge (A \vee C) \rightarrow A \vee (B \wedge C)$

Rules of inference;

- (R1) B is deduced from A and $A \rightarrow B$ (modus ponens, MP);
- (R2) $A \rightarrow C$ is done from $A \rightarrow B$ and $B \rightarrow C$;
- (R3) $A \rightarrow B \wedge C$ from $A \rightarrow B$ and $A \rightarrow C$;
- (R4) $B \vee C \rightarrow A$ from $B \rightarrow A$ and $C \rightarrow A$;
- (R5) $\neg A \rightarrow \neg B$ from $B \rightarrow A$;
- (R6) $(B \rightarrow C) \rightarrow (A \rightarrow C)$ from $A \rightarrow B$;
- (R7) $(C \rightarrow A) \rightarrow (C \rightarrow B)$ from $A \rightarrow B$ and $B \rightarrow A$.

Let A be a formula of ML . By $\vdash_{ML} A$ we mean that there is a sequence of formulas A_1, A_2, \dots, A_n of ML such that

- (1) $A = A_n$
- (2) For every A_i , it is an axiom or it is deduced from A_j and A_k ($j, k < i$) by rules of inference.

We say that A is provable in ML when $\vdash_{ML} A$. If no confusion arises, we denote it simply by $\vdash A$.

It is easy to show the next lemmas, so we omit their proofs.

Lemma 5. *For every formula A and B of ML , we have that $\vdash (A \rightarrow A) \rightarrow (B \rightarrow B)$.*

We denote $p_1 \rightarrow p_1$ by t . Thus it is clear that $\vdash t \rightarrow (A \rightarrow A)$ for every formula A .

A function $\tau : \Pi \longrightarrow \mathbf{M}$ is called a valuation function. The domain of the valuation function can be extended uniquely to the set Φ of all formulas of ML as follows:

$$\begin{aligned}\tau(A \wedge B) &= \tau(A) \wedge \tau(B) \\ \tau(A \vee B) &= \tau(A) \vee \tau(B) \\ \tau(\neg A) &= N\tau(A) \\ \tau(A \rightarrow B) &= \tau(A) \rightarrow \tau(B)\end{aligned}$$

Henceforth we use the same symbol τ for the extended valuation function.

We can show that the de Morgan logic ML is sound for the MI -algebra \mathbf{M} , that is, if $\vdash_{ML} A$ then $\tau(A) = 1$ for any valuation function τ .

Theorem 2. *Let A be an arbitrary formula of ML . If $\vdash_{ML} A$ then $\tau(A) = 1$ for every valuation function τ .*

Proof. By induction on the construction of a proof. It suffices to show that $\tau(X) = 1$ for every axiom X and that the rules of inference preserve the validity.

As corollaries to the theorem we have the following.

Corollary 1. *ML is consistent.*

Proof. Since $\tau(\neg t) = 0$, the formula $\neg t$ is not provable in ML . Thus the de Morgan logic is consistent.

Corollary 2. *The de Morgan logic ML is different from the classical propositional logic (CPL), the intuitionistic propositional logic (IPL), and Kleene logic (KL).*

Proof. If we think of a valuation function τ such that $\tau(p_1) = a$ and $\tau(p_n) = b$ for $n = 2, 3, \dots$, then we have that $\tau(p_1 \wedge \neg p_1) = a$ and $\tau(p_2 \vee \neg p_2) = b \neq 1$ and hence that the formulas $p_2 \vee \neg p_2$ and $(p_1 \wedge \neg p_1) \rightarrow (p_2 \vee \neg p_2)$ are not provable in ML . Thus ML is different from CPL and from KL . Next, the formula $\neg\neg A \rightarrow A$ is not provable in IPL in general, but it is provable in ML .

4. COMPLETENESS THEOREM

In this section we shall establish the completeness theorem of the de Morgan logic ML , and it is the main theorem of this paper. The completeness theorem of ML means that a formula A is provable in ML if $\tau(A) = 1$ for any valuation function τ . As a method to show the theorem, we consider the Lindenbaum algebra of ML and investigate the property of that algebra.

We introduce the relation \equiv on Φ as follows. For $A, B \in \Phi$,

$$A \equiv B \text{ iff } \vdash_{ML} A \rightarrow B \text{ and } \vdash_{ML} B \rightarrow A.$$

Lemma 6. *The relation \equiv is a congruent relation on Φ .*

Proof. We only show that the relation \equiv satisfies the next conditions: If $A \equiv X$ and $B \equiv Y$, then

- (a) $(A \wedge B) \equiv (X \wedge Y)$
- (b) $(A \vee B) \equiv (X \vee Y)$
- (c) $\neg A \equiv \neg X$
- (d) $(A \rightarrow B) \equiv (X \rightarrow Y)$

It is evident that the conditions (a), (b), and (c) hold from axioms (A2), (A2)', (A3), (A3)', and the rules of inference (R2), (R5), (R6). We prove that the condition (d) holds.

Suppose that $A \equiv X$ and $B \equiv Y$. Since $\vdash X \rightarrow A$, we have $\vdash (A \rightarrow B) \rightarrow (X \rightarrow B)$ by (R6). On the other hand, since $\vdash B \rightarrow Y$ and $\vdash Y \rightarrow B$, it follows that $(X \rightarrow B) \rightarrow (X \rightarrow Y)$ by (R7) and hence that $\vdash (A \rightarrow B) \rightarrow (X \rightarrow Y)$. Similarly it follows that $\vdash (X \rightarrow Y) \rightarrow (A \rightarrow B)$.

Hence the relation \equiv is the congruent relation.

We think of the property of the quotient set Φ/\equiv by the congruent relation \equiv . We set $\Phi/\equiv = \{[A] \mid A \in \Phi\}$, where $[A] = \{X \in \Phi \mid A \equiv X\}$. We introduce an order relation \sqsubseteq on Φ/\equiv as follows: For any $[A], [B] \in \Phi/\equiv$,

$$[A] \sqsubseteq [B] \text{ iff } \vdash_{ML} A \rightarrow B.$$

Since the relation \equiv is congruent, it is clear that the definition of \sqsubseteq is well-defined and that the relation \sqsubseteq is a partial order. Concerning to this order we have

Lemma 7. $\inf\{[A], [B]\} = [A \wedge B]$, $\sup\{[A], [B]\} = [A \vee B]$

Proof. We shall show the first case for the sake of simplicity. The second case can be proved analogously. Since $\vdash (A \wedge B) \rightarrow A$ and $\vdash (A \wedge B) \rightarrow B$, we obtain $[A \wedge B] \sqsubseteq [A]$, $[A \wedge B] \sqsubseteq [B]$. For any $[C]$ such that $[C] \sqsubseteq [A]$, $[B]$, since $\vdash C \rightarrow A$ and $\vdash C \rightarrow B$, it follows that $\vdash C \rightarrow (A \wedge B)$ by (R3). Thus we have $\inf\{[A], [B]\} = [A \wedge B]$.

By the lemma we can define the operations \sqcap and \sqcup by

$$\begin{aligned} [A] \sqcap [B] &= \inf\{[A], [B]\} = [A \wedge B] \\ [A] \sqcup [B] &= \sup\{[A], [B]\} = [A \vee B]. \end{aligned}$$

It is easy to show that the structure $(\Phi/\equiv; \sqcap, \sqcup)$ is a lattice. Moreover, if we put $[t] = \mathbf{1}$, $\mathbf{N}[A] = [\neg A]$, and $[A] \Rightarrow [B] = [A \rightarrow B]$, then the axioms of ML assures that the structure $(\Phi/\equiv; \sqcap, \sqcup, \mathbf{N}, \Rightarrow, \mathbf{0}, \mathbf{1})$ is a MI -algebra. The structure is called a Lindenbaum algebra of ML . Hence we have the theorem.

Theorem 3. *The Lindenbaum algebra Φ/\equiv of the de Morgan logic ML is the MI -algebra.*

As to that algebra Φ/\equiv , we have an important lemma.

Lemma 8. *For every formula A , $\vdash_{ML} A$ iff $[A] = \mathbf{1}$ in Φ/\equiv .*

Proof. Suppose that $\vdash A$. Since $A \rightarrow (t \rightarrow A)$ is provable in ML , we get that $\vdash t \rightarrow A$, that is, $[A] = \mathbf{1}$. Conversely if we assume that $[A] = \mathbf{1}$ then it follows $\vdash t \rightarrow A$ by definition. Thus we have $\vdash A$ by $\vdash t$.

Now we shall prove the completeness theorem of ML . In order to show that, it suffices to indicate the existence of a valuation function τ such that $\tau(A) \neq \mathbf{1}$ if A is not provable in ML . Suppose that a formula A is not provable in ML . In the Lindenbaum algebra Φ/\equiv of ML , we have $[A] \neq \mathbf{1}$ by the lemma above. It means that $\mathbf{N}[A] \neq \mathbf{0}$. By lemma 1, there is a maximal filter M in Φ/\equiv such that $\mathbf{N}[A] \in M$. By theorem 1, there exists a quotient MI -algebra $(\Phi/\equiv)^\sharp$ which is embeddable to \mathbf{M} by $\xi : (\Phi/\equiv)^\sharp \rightarrow \mathbf{M}$, where $\xi([X]^\sharp) = j$ if $[X] \in (\Phi/\equiv)_j$. For any propositional variable p , we put $\tau(p) = j$ if $[p] \in (\Phi/\equiv)_j$, that is,

$$\tau(p) = \begin{cases} 1 & \text{if } [p] \in M, \mathbf{N}[p] \notin M \\ a & \text{if } [p] \in M, \mathbf{N}[p] \in M \\ b & \text{if } [p] \notin M, \mathbf{N}[p] \notin M \\ 0 & \text{if } [p] \notin M, \mathbf{N}[p] \in M. \end{cases}$$

As to that function τ , we can show the next lemma.

Lemma 9. *For any formula $X \in \Phi$,*

$$\tau(X) = \begin{cases} 1 & \text{if } [X] \in M, \mathbf{N}[X] \notin M \\ a & \text{if } [X] \in M, \mathbf{N}[X] \in M \\ b & \text{if } [X] \notin M, \mathbf{N}[X] \notin M \\ 0 & \text{if } [X] \notin M, \mathbf{N}[X] \in M. \end{cases}$$

Proof. It suffices to show that τ satisfies the followings: For arbitrary formulas X and Y ,

- (a) $\tau(X \wedge Y) = \tau(X) \wedge \tau(Y)$
- (b) $\tau(X \vee Y) = \tau(X) \vee \tau(Y)$
- (c) $\tau(\neg X) = \mathbf{N}\tau(X)$
- (d) $\tau(X \rightarrow Y) = \tau(X) \rightarrow \tau(Y)$

We only show the case (d). Let $x = [X]$ and $y = [Y]$.

Case (d):

• $\tau(X) = a, \tau(Y) = 0$: It is sufficient to prove that $x \Rightarrow y \in L_a$, that is, $x \Rightarrow y$ and $\mathbf{N}(x \Rightarrow y)$ are in M . By assumption we have $x \in M, \mathbf{N}x \in M, y \notin M, \mathbf{N}y \in M$. From (I3) $\mathbf{N}x \sqcap \mathbf{N}y \leq (x \Rightarrow y) \sqcup y$, since $y \notin M$, we have $x \Rightarrow y \in M$. Next, it follows from (I7) that $\mathbf{N}(x \Rightarrow y) \sqcup y \in M$ and hence that $\mathbf{N}(x \Rightarrow y) \in M$ because of $y \notin M$. These imply that $x \Rightarrow y \in L_a$.

• $\tau(X) = a, \tau(Y) = b$: In this case we have $x, \mathbf{N}x \in M$ and $y, \mathbf{N}y \notin M$ by definition. We can conclude that $x \Rightarrow y \notin M$, because (I5) $x \sqcap (x \Rightarrow y) \leq y \sqcup \mathbf{N}y \notin M$ and $x \in M$. That $\mathbf{N}y \notin M$ yields $\mathbf{N}(x \Rightarrow y) \notin M$ by (I1).

The other cases are proved similarly.

Well, since $\mathbf{N}[A] \in M$, it follows that $\tau(A) \neq \mathbf{1}$ by that lemma. Hence we have the completeness theorem of ML .

Theorem 4. *For any formula A , $\vdash_{ML} A$ iff $\tau(A) = 1$ for every valuation function τ .*

It turns out from the theorem that it is sufficient to calculate the value of $\tau(A)$ for every valuation τ , in order to show whether a formula A is provable or not in ML . Since any formula has at most finite numbers of propositional variables, say n , the possible values of the n -tuple of the propositional variables of that formula are finite (at most 4^n). Thus we can establish that

Theorem 5. *The de Morgan logic ML is decidable.*

REFERENCES

- [1] A. Bialynicki-Birula, "Remarks on quasi-Boolean algebras," Bull. Ac. Pol. Sc, vol.5 (1957), pp. 615-619.
- [2] M. Dunn, "Relevance logic and entailment," in Handbook of Philos. Logic vol 3, pp. 117-224, D.Gabbay and F. Guenther eds., D. Reidel, 1987.
- [3] M.Kondo, "Completeness theorem of Kleene logic," Mem. Fac. Sci. Shimane Univ., vol.29 (1995), pp. 77-87.
- [4] J.Lukasiewicz, "On three-valued logic," in Polish Logic, McCall, ed., Clarendon Press, Oxford, 1967.
- [5] H. Rasiowa, An Algebraic Approach to Non-Classical Logics, North-Holland, Amsterdam, 1974.

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