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CAPACITY AND HAUSDORFF CONTENT OF CERTAIN ENLARGED SETS

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Abstract. We consider certain enlargement operations of sets which are related to the Nagel-Stein approach regions and the maximal functions. The capacity and the Hausdorff content of enlarged sets are estimated and their relationship is shown. These results are applied to the boundary behavior of harmonic functions in the upper half space.

1. Introduction

Let Ω be a set in $\mathbb{R}^{n+1}_+$ with $\overline{\Omega} \cap \partial\mathbb{R}^{n+1}_+ = \{0\}$. For any measurable function u on $\mathbb{R}^{n+1}_+$ we define the maximal function by $M_\Omega u(x) = \sup_{x+\Omega} |u|$. In particular, if Ω is the nontangential cone $\Gamma = \{(x, y) : |x| < y\}$, then we write Nu for $M_\Omega u$. This is a nontangential maximal function. We say that Ω satisfies a cone condition with aperture $\alpha > 0$ if

$$(x_1, y_1) \in \Omega \text{ and } |x - x_1| < \alpha(y - y_1) \implies (x, y) \in \Omega.$$

Put $\Omega(y) = \{x : (x, y) \in \Omega\}$. We say that Ω satisfies a cross section condition if

$$|\Omega(y)| \le Ay^n,$$

where $|\Omega(y)|$ denotes the Lebesgue measure of $\Omega(y)$ and A is an absolute positive constant whose value is unimportant and may change from line to line. If Ω satisfies both of the cone condition and the cross section condition, then we say Ω satisfies the Nagel-Stein condition (abbreviated to (NS)). Andersson and Carlsson [7] proved

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Theorem A. *Let Ω satisfy (NS). Then*

$$|\{x \in \mathbb{R}^n : M_\Omega u(x) > \lambda\}| \le A|\{x \in \mathbb{R}^n : Nu(x) > \lambda\}| \tag{1.1}$$

for any measurable function u on $\mathbb{R}^{n+1}_+$ and $\lambda > 0$, whence

$$\|M_\Omega u\|_p \le A\,\|Nu\|_p \quad \text{for } p > 0.$$

A similar and essentially equivalent result has been already observed by Sueiro [20]. Theorem A has an application to the boundary behavior of harmonic functions. For a locally integrable function f on $\mathbb{R}^n$ we let Pf be the Poisson integral of f. Letting $u = Pf$, we obtain from Theorem A and the classical maximal inequality

$$\|M_\Omega(Pf)\|_p \le A\,\|f\|_p \quad \text{for } p > 1 \tag{1.2}$$

and $|\{x \in \mathbb{R}^n : M_\Omega Pf(x) > \lambda\}| \le A\,\|f\|_1/\lambda$ (Nagel and Stein [18, Theorem 2]). Theorem A can also be applied to the local Fatou theorem (Svensson [21]). See also [14].

In (1.1) the Lebesgue measure of two sets are compared. In this paper, we shall prove inequalities similar to (1.1). In these inequalities we shall compare capacity and Hausdorff content of the above sets.

Let $K(r) \not\equiv 0$ be a nonnegative nonincreasing lower semicontinuous function for $r > 0$. For $x \in \mathbb{R}^n$ we define $K(x) = K(|x|)$, and assume that $K(x)$ is locally integrable on $\mathbb{R}^n$. Let $p > 1$. For $E \subset \mathbb{R}^n$ we define the capacity $C_{K,p}(E)$ by

$$C_{K,p}(E) = \inf\{\|f\|_p^p : f \ge 0,\ K * f \ge 1 \text{ on } E\}.$$

Let $h(r)$ be a positive nondecreasing function for $r > 0$ and $h(0) = 0$. Such a function is called a measure function. We define the content Λ_h by

$$\Lambda_h(E) = \inf\{\sum h(r_j) : E \subset \bigcup B(x_j, r_j)\},$$

where $B(x,r)$ stands for the n-dimensional open ball with center at x and radius r. If $h(r) = r^\beta$, then we write Λ_β for Λ_h and call it the β-dimensional Hausdorff content. We assume that

$$h(r) \ge Ar^n \quad \text{for } 0 < r < 1, \tag{1.3}$$

since otherwise $\Lambda_h(E) = 0$ for any bounded set E.

Theorem 1. *Let Ω satisfy (NS).*

(i) *For any measurable function u on $\mathbb{R}^{n+1}_+$ and $\lambda > 0$*

$$C_{K,p}(\{x \in \mathbb{R}^n : M_\Omega u(x) > \lambda\}) \le AC_{K,p}(\{x \in \mathbb{R}^n : Nu(x) > \lambda\}).$$

(ii) *Suppose h satisfies*

$$h(r)h(t) \le Ah(rt) \quad \text{for } r > 0 \text{ and } 0 < t < 1. \tag{1.4}$$

Then for any measurable function u on $\mathbb{R}^{n+1}_+$ and $\lambda > 0$

$$\Lambda_h(\{x \in \mathbb{R}^n : M_\Omega u(x) > \lambda\}) \le A\Lambda_h(\{x \in \mathbb{R}^n : Nu(x) > \lambda\}).$$

Here we note that if (1.4) holds then, h satisfies the doubling condition

$$h(r) \le h(2r) \le Ah(r) \quad \text{for } r > 0. \tag{1.5}$$

In our previous papers [5] and [4] we have considered certain expansions of sets and their capacity and Hausdorff content. We shall observe that the above result concerning maximal functions follows from such consideration. For $E \subset \mathbb{R}^n$ we put

$$\widetilde{E}_\Omega = \bigcup_{x \in E} (x - \Omega(\delta_E(x))),$$

where $\delta_E(x) = \operatorname{dist}(x, E^c)$. Theorem 1 will be deduced from the following theorem.

Theorem 2. *Let Ω satisfy (NS).*

(i) *We have $C_{K,p}(\widetilde{E}_\Omega) \le AC_{K,p}(E)$ for any $E \subset \mathbb{R}^n$.*
(ii) *Suppose h satisfies (1.4). Then $\Lambda_h(\widetilde{E}_\Omega) \le A\Lambda_h(E)$ for any $E \subset \mathbb{R}^n$*

We note that Theorems 1 and 2 have partial converse.

Proposition 1. *Let h satisfy (1.3) and (1.4) and let Ω satisfy the cone condition. Suppose one of the following conditions holds:*

(i) $\Lambda_h(\{x \in \mathbb{R}^n : M_\Omega u(x) > \lambda\}) \le A\Lambda_h(\{x \in \mathbb{R}^n : Nu(x) > \lambda\})$ *for any u,*
(ii) $\Lambda_h(\widetilde{E}_\Omega) \le A\Lambda_h(E)$ *for any $E \subset \mathbb{R}^n$.*

Then Ω satisfies the cross section condition.

In general, we cannot replace the Hausdorff content Λ_h by the capacity $C_{K,p}$ in the assumptions (i) and (ii) in Proposition 1. For special capacities, however, we can replace them. Let $k_\alpha(t) = t^{\alpha-n}$ be the Riesz kernel. The capacity $C_{K,p}$ with respect to $K = k_\alpha$ is called the Riesz capacity of index (α, p) and denoted by $R_{\alpha,p}$.

Proposition 2. *Let $\alpha p < n$ and let let Ω satisfy the cone condition. Suppose one of the following conditions holds:*

(i) $R_{\alpha,p}(\{x \in \mathbb{R}^n : M_\Omega u(x) > \lambda\}) \le AR_{\alpha,p}(\{x \in \mathbb{R}^n : Nu(x) > \lambda\})$ *for any u,*
(ii) $R_{\alpha,p}(\widetilde{E}_\Omega) \le AR_{\alpha,p}(E)$ *for any $E \subset \mathbb{R}^n$.*

Then Ω satisfies the cross section condition.

We can consider "tangential" extension. Let $\eta(t)$ be a positive nondecreasing function for $t > 0$ such that $\eta(0) = 0$. Let

$$\Gamma^\eta = \{(x, y) : x \in B(0, \eta(y))\} \quad \text{and} \quad \Omega^\eta = \{(x, y) : x \in \Omega(\eta(y))\}.$$

It is easy to see that $\Gamma^\eta = \{(x, y) : |x| < \eta(y)\}$, and so Γ^η is "tangential" if $\eta(y)/y \to \infty$ as $y \to 0$. Let

$$N^\eta u(x) = \sup_{x+\Gamma^\eta} |u| \quad \text{and} \quad M_{\Omega^\eta} u(x) = \sup_{x+\Omega^\eta} |u|.$$

These are tangential maximal functions. The following theorem is a generalization of Theorem 1. When $\eta(y) = y$, the theorem is identical with Theorem 1.

Theorem 3. *Let Ω satisfy (NS).*

(i) *For any measurable function u on $\mathbb{R}^{n+1}_+$ and $\lambda > 0$*

$$C_{K,p}(\{x \in \mathbb{R}^n : M_{\Omega^\eta} u(x) > \lambda\}) \le A C_{K,p}(\{x \in \mathbb{R}^n : N^\eta u(x) > \lambda\}).$$

(ii) *Suppose h satisfies (1.4). Then for any measurable function u on $\mathbb{R}^{n+1}_+$ and $\lambda > 0$*

$$\Lambda_h(\{x \in \mathbb{R}^n : M_{\Omega^\eta} u(x) > \lambda\}) \le A \Lambda_h(\{x \in \mathbb{R}^n : N^\eta u(x) > \lambda\}).$$

Let us consider tangential enlargement of sets. Put

$$\widetilde{E}^\eta = \bigcup_{x \in E} B(x, \eta(\delta_E(x))) \quad \text{and} \quad \widetilde{E}_{\Omega^\eta} = \bigcup_{x \in E} (x - \Omega(\eta(\delta_E(x)))) .$$

Theorem 4. *Let Ω satisfy (NS).*

(i) *We have $C_{K,p}(\widetilde{E}_{\Omega^\eta}) \le A C_{K,p}(\widetilde{E}^\eta)$ for any $E \subset \mathbb{R}^n$.*

(ii) *Suppose h satisfies (1.4). Then $\Lambda_h(\widetilde{E}_{\Omega^\eta}) \le A \Lambda_h(\widetilde{E}^\eta)$ for any $E \subset \mathbb{R}^n$.*

We note that Theorem 4 is not a generalization of Theorem 2. When $\eta(y) = y$, Theorem 4 gives inequalities weaker than those in Theorem 2. Both Theorems 3 and 4 will be proved as corollaries to Theorem 2.

Now let us compare the capacity and the Hausdorff content. Hereafter, let $1/p + 1/q = 1$. In order to avoid the trivial case, we assume that

$$\int_r^\infty K(t)^q t^n \frac{dt}{t} < \infty \quad \text{for } r > 0.$$

Let

$$\overline{K}(r) = \frac{1}{r^n} \int_0^r K(t) t^n \frac{dt}{t} \quad \text{and} \quad \Phi(r) = \left(\int_r^\infty \overline{K}(t)^q t^n \frac{dt}{t} \right)^{1-p}.$$

In general, we write $f \approx g$ if there is $A \ge 1$ such that $A^{-1} f \le g \le A f$. In [3], we have shown the following theorem.

Theorem B. *Let Φ be as above. Then $C_{K,p}(B(x,r)) \approx \Phi(r)$ for $r > 0$. In particular, $C_{K,p}(E) \le A \Lambda_\Phi(E)$ for any $E \subset \mathbb{R}^n$. Conversely, if a measure function h satisfies*

$$\int_0^\infty h(t)^{q-1} \overline{K}(t)^q t^n \frac{dt}{t} < \infty, \tag{1.6}$$

then $\Lambda_h(E) \le A C_{K,p}(E)$ for any $E \subset \mathbb{R}^n$.

Let us generalize this theorem. A generalization of the first assertion is as follows.

Theorem 5. *Let $h(t)$ satisfy (1.4), $\eta(t) \geq t$ and $\Phi(\eta(t)) \leq Ah(t)$. Then $C_{K,p}(\widetilde{E}^\eta) \leq A\Lambda_h(E)$ for any $E \subset \mathbb{R}^n$.*

The opposite direction is more interesting. The following theorem is a generalization of [4, Theorems 2 and 2'].

Theorem 6. *Let $h(t)$ satisfy (1.5), $\eta(t) \geq t$ and $h(\eta(t)) \leq A\Phi(t)$. Moreover, suppose*

$$\int_0^1 h(\eta(r)t)^{q-1}\overline{K}(rt)^q(rt)^n \frac{dt}{t} \leq A \quad \text{for } r > 0 \tag{1.7}$$

Then $\Lambda_h(\widetilde{E}^\eta) \leq AC_{K,p}(E)$ for any $E \subset \mathbb{R}^n$.

By the change of the variable it is easy to see that (1.7) implies (1.6). Theorems 5 and 6 apply to maximal functions.

Corollary 1. *Let $h(t)$ satisfy (1.4), $\eta(t) \geq t$ and $\Phi(\eta(t)) \leq Ah(t)$. Suppose u is a measurable function on $\mathbb{R}^{n+1}_+$ and $\lambda > 0$. Then*

$$C_{K,p}(\{x \in \mathbb{R}^n : N^\eta u(x) > \lambda\}) \leq A\Lambda_h(\{x \in \mathbb{R}^n : Nu(x) > \lambda\}).$$

Moreover, if Ω satisfies (NS), then

$$C_{K,p}(\{x \in \mathbb{R}^n : M_{\Omega^\eta}u(x) > \lambda\}) \leq A\Lambda_h(\{x \in \mathbb{R}^n : Nu(x) > \lambda\}).$$

Corollary 2. *Let $h(t)$ satisfy (1.5), $\eta(t) \geq t$, $h(\eta(t)) \leq A\Phi(t)$ and (1.7) hold. Suppose u is a measurable function on $\mathbb{R}^{n+1}_+$ and $\lambda > 0$. Then*

$$\Lambda_h(\{x \in \mathbb{R}^n : N^\eta u(x) > \lambda\}) \leq AC_{K,p}(\{x \in \mathbb{R}^n : Nu(x) > \lambda\}).$$

Moreover, if Ω satisfies (NS) and h satisfies (1.4), then

$$\Lambda_h(\{x \in \mathbb{R}^n : M_{\Omega^\eta}u(x) > \lambda\}) \leq AC_{K,p}(\{x \in \mathbb{R}^n : Nu(x) > \lambda\}).$$

A different type of the comparison can be proved. The following result is obtained in a joint work with M. Mizuta. This is also used for the proof of Proposition 2.

Theorem 7. *Suppose h is strictly increasing and*

$$\int_0^r \left(\frac{h(t)}{h(r)}\right)^{q-1} \overline{K}(t)^q t^n \frac{dt}{t} \leq A \int_r^\infty \overline{K}(t)^q t^n \frac{dt}{t}. \tag{1.8}$$

Then for any set $E \subset \mathbb{R}^n$

$$h^{-1}(\Lambda_h(E)) \leq \Phi^{-1}(AC_{K,p}(E)).$$

In particular, if $n - \alpha p < \beta \le n$, then

$$\Lambda_\beta(E)^{1/\beta} \le AR_{\alpha,p}(E)^{1/n-\alpha p}.$$

The above results have applications to norm estimates of maximal functions and the boundary behavior of harmonic functions. For a function f on $\mathbb{R}^n$ and $r > 0$ we define the integral with respect to the capacity $C_{K,p}$ and the Hausdorff content Λ_h by

$$\int |f|^r dC_{K,p} = \int_0^\infty C_{K,p}(\{x : |f(x)| > \lambda\}) d\lambda^r,$$
$$\int |f|^r d\Lambda_h = \int_0^\infty \Lambda_h(\{x : |f(x)| > \lambda\}) d\lambda^r.$$

The following results are almost immediate from Theorems 1–6.

Corollary 3. *Let Ω satisfy (NS) and $r > 0$. Suppose u is a measurable function on $\mathbb{R}^{n+1}_+$.*

(i) $\int (M_{\Omega^\eta} u)^r dC_{K,p} \le A \int (N^\eta u)^r dC_{K,p}$.

(ii) *If h satisfies (1.4), then*

$$\int (M_{\Omega^\eta} u)^r d\Lambda_h \le A \int (N^\eta u)^r d\Lambda_h.$$

(iii) *If h and η satisfy the conditions in Theorem 5, then*

$$\int (M_{\Omega^\eta} u)^r dC_{K,p} \le A \int (N u)^r d\Lambda_h.$$

(iv) *If h and η satisfy the conditions in Theorem 6, then*

$$\int (M_{\Omega^\eta} u)^r d\Lambda_h \le A \int (N u)^r dC_{K,p}.$$

Observe that $N(P(K * f)) \le K * N(Pf)$, so that the capacity strong type inequality ([10]) yields

$$\int (N(P(K * f)))^p dC_{K,p} \le \int C_{K,p}(K * N(Pf) > \lambda) d\lambda^p \le A \left\| N(Pf) \right\|_p^p \le A \left\| f \right\|_p^p,$$

where the last inequality follows from the usual maximal inequality. Thus the above theorem has the following corollaries concerning the Poisson integral $P(K * f)$.

Corollary 4. *Let Ω satisfy (NS). Then for any measurable function f on $\mathbb{R}^n$*

$$\int (M_\Omega(P(K*f)))^p dC_{K,p} \leq A\|f\|_p^p .$$

Moreover, if h and η satisfy the conditions in Theorem 6, then

$$\int (M_{\Omega^\eta}(P(K*f)))^p d\Lambda_h \leq A\|f\|_p^p .$$

In the standard way we can obtain the following result concerning the boundary behavior of harmonic functions.

Corollary 5. *Let Ω satisfy (NS). Suppose $\|f\|_p < \infty$. Then there is a set $E \subset \mathbb{R}^n$ such that $C_{K,p}(E) = 0$ and for $\xi \in \mathbb{R}^n \setminus E$*

$$\lim_{\substack{(x,y)\to\xi \\ (x,y)\in\xi+\Omega}} P(K*f)(x,y) = K*f(\xi).$$

Moreover, if h and η satisfy the conditions in Theorem 6, then there is a set $F \subset \mathbb{R}^n$ such that $\Lambda_h(F) = 0$ and for $\xi \in \mathbb{R}^n \setminus F$

$$\lim_{\substack{(x,y)\to\xi \\ (x,y)\in\xi+\Omega^\eta}} P(K*f)(x,y) = K*f(\xi).$$

The above corollaries generalize some results of [17], [2] and [4]. Corollary 3 (ii) applies to the boundary behavior of Poisson integral of Sobolev and Besov functions. Following [9] we introduce some notation. Assume that for each ball $B \subset \mathbb{R}^n$ there exists a linear operator $\mathcal{A}_B : L^1_{loc}(\mathbb{R}^n) \to C(\mathbb{R}^n)$ with the following properties:

(i) There is a constant $A > 0$ such that for any ball B and any $x \in B$

$$|\mathcal{A}_B(x)| \leq A\frac{1}{|B|}\int_B |f|dy.$$

(ii) If $B' \subset B$, then $\mathcal{A}_{B'}(\mathcal{A}_B f) = \mathcal{A}_B f$. If λ is a constant, then $\mathcal{A}_B(\lambda) = \lambda$.

Such a family is called a family of averaging operators and denoted by $\mathcal{A}$. For $a > 0$ we define

$$f^\#_{a,\mathcal{A}}(x) = f^\#_a(x) = \sup_{x\in B} r(B)^{-a}\frac{1}{|B|}\int_B |f - \mathcal{A}_B f|dy,$$

where $r(B)$ stands for the radius of B. For $1 \leq p \leq \infty$ we let $C^{p,a}_{\mathcal{A}}(\mathbb{R}^n) = C^{p,a}(\mathbb{R}^n)$ be the space of those L^p functions f such that $\|f^\#_a\|_p < \infty$, endowed with the norm $\|f\|_{p,a} = \|f\|_p + \|f^\#_a\|_p$. Then [9, Theorem 3.2] for the Euclidean case can be read as follows:

Theorem C. *Let* $1 < p < n/a$, $n - ap < \beta \le n$, $\gamma = (n-ap)/\beta$ *and* $\eta(t) = \max\{ct^\gamma, t\}$ *with* $c > 0$. *If* $f \in C^{p,a}(\mathbb{R}^n)$ *and* u *is its Poisson integral, then*

$$\int (N^\eta u)^p d\Lambda_\beta \le A \|f\|_{p,a}^p .$$

With the aid of Corollary 3 (ii) we can generalize Theorem C without any extra effort.

Corollary 6. *Let* p, a, β, γ *and* η *be as above. Let* Ω *satisfy (NS). If* $f \in C^{p,a}(\mathbb{R}^n)$ *and* u *is its Poisson integral, then*

$$\int (M_{\Omega^\eta} u)^p d\Lambda_\beta \le A \|f\|_{p,a}^p .$$

Of course, the above corollary implies the existence of the limit of u along $x + \Omega^\eta$ for Λ_β-a.e. $x \in \mathbb{R}^n$.

2. Proof of Theorem 1 (i) and Theorem 2 (i)

In this section we shall first prove Theorem 2 (i) and use it to prove Theorem 1(i). For $E \subset \mathbb{R}^n$ we let $T(E) = \{(x,y) : B(x,y) \subset E\}$. This is a "tent" over E. For $\mathcal{E} \subset \mathbb{R}^{n+1}_+$ we let $\mathcal{E}^* = \bigcup_{(x,y)\in\mathcal{E}} B(x,y)$. This is a kind of "projection" of $\mathcal{E}$ onto $\mathbb{R}^n$. We immediately obtain the following relationship between $T(E)$ and $\mathcal{E}^*$.

Lemma 1. *Suppose* $E \subset \mathbb{R}^n$. *Then* $(T(E))^* \subset E$. *Moreover, if* E *is an open set then* $(T(E))^* = E$.

It is easy to see that if Ω satisfies the cone condition, then $\Omega(y)$ is a nondecreasing set function of y, i.e., $\Omega(y_1) \subset \Omega(y_2)$ if $0 < y_1 < y_2$.

Lemma 2. *Let* $\Omega(y)$ *be a nondecreasing set function of* y. *Suppose* $\mathcal{E} \subset \mathbb{R}^{n+1}_+$. *Then*

$$\bigcup_{(x,y)\in\mathcal{E}} (x - \Omega(y)) \subset \widetilde{E}_\Omega \quad \text{with } E = \mathcal{E}^*.$$

Proof. Let ξ be a point in the left hand side. Then there is a point $(x,y) \in \mathcal{E}$ such that $\xi \in x - \Omega(y)$. By the definition of $\mathcal{E}^*$ we have $B(x,y) \subset \mathcal{E}^* = E$, and hence $y \le \delta_E(x)$. Since $\Omega(y)$ is nondecreasing, it follows that $\xi \in x - \Omega(\delta_E(x))$, so that $\xi \in \widetilde{E}_\Omega$.

Lemma 3. *Suppose* $E \subset \mathbb{R}^n$ *and let*

$$\widehat{E}_\Omega = \{\xi \in \mathbb{R}^n : (\xi + \Omega) \cap T(E) \ne \emptyset\}.$$

Then $\widetilde{E}_\Omega \subset \widehat{E}_\Omega$*. Moreover, if* $\Omega(y)$ *is a nondecreasing set function of* y*, then* $\widetilde{E}_\Omega = \widehat{E}_\Omega$*.*

Proof. Suppose $\xi \in \widetilde{E}_\Omega$. Then there is $x \in E$ such that $\xi \in x - \Omega(\delta_E(x))$, or equivalently $(x, \delta_E(x)) \in \xi + \Omega$. By definition $B(x, \delta_E(x)) \subset E$, whence $(x, \delta_E(x)) \in T(E)$. Hence $(x, \delta_E(x)) \in (\xi + \Omega) \cap T(E)$ and $\xi \in \widehat{E}_\Omega$. Thus $\widetilde{E}_\Omega \subset \widehat{E}_\Omega$. Now suppose that $\Omega(y)$ is a nondecreasing set function of y. We observe that

$$\widehat{E}_\Omega = \bigcup_{(x,y)\in T(E)} (x - \Omega(y)).$$

Hence Lemmas 1 and 2 yield $\widehat{E}_\Omega \subset \widetilde{E}_\Omega$. The lemma follows.

Proof of Theorem 2 (i). Let f be an nonnegative function such that $K * f \ge 1$ on E. Suppose $(x, y) \in T(E)$. Then $B(x, y) \subset E$ and hence $P(K * f)(x, y) \ge P(\chi_{B(x,y)})(x, y) = A_0$ with A_0 depending only on the dimension. By definition $M_\Omega(P(K*f)) > A_0$ on $\widehat{E}_\Omega$ and hence on $\widetilde{E}_\Omega$ by Lemma 3. Moreover, by inspection, $M_\Omega(P(K*f)) \le K * M_\Omega(Pf)$. Hence $M_\Omega(Pf)/A_0$ is a test function for $C_{K,p}(\widetilde{E}_\Omega)$. Therefore

$$C_{K,p}(\widetilde{E}_\Omega) \le \|M_\Omega(Pf)/A_0\|_p^p \le A \,\|f\|_p^p \,,$$

where the second inequality follows from (1.2). Taking the infimum with respect to f, we obtain $C_{K,p}(\widetilde{E}_\Omega) \le A C_{K,p}(E)$. The theorem follows.

Remark. In the above proof we need the strong (p, p) for $M_\Omega(Pf)$, which is valid only for $p > 1$. The above proof does not work for $p = 1$. The case $p = 1$ will be treated later.

Now let us prove Theorem 1 (i). In the proof we may assume that $u \ge 0$. Let us begin with

Lemma 4. *Let* u *be a nonnegative function on* $\mathbb{R}^{n+1}_+$ *and* $\mathcal{E} = \{(x, y) : u(x, y) > \lambda\}$ *for* $\lambda > 0$*. Then*

(i) $\{x \in \mathbb{R}^n : Nu(x) > \lambda\} = \bigcup_{(x,y)\in\mathcal{E}} B(x, y)$,
(ii) $\{x \in \mathbb{R}^n : M_\Omega u(x) > \lambda\} = \bigcup_{(x,y)\in\mathcal{E}} (x - \Omega(y))$.

Proof. We prove only (ii). Observe

$$\begin{aligned} M_\Omega u(x) > \lambda &\iff \exists (x_1, y_1) \in x + \Omega \text{ such that } u(x_1, y_1) > \lambda \\ &\iff x \in x_1 - \Omega(y_1) \text{ with } (x_1, y_1) \in \mathcal{E}. \end{aligned}$$

The lemma follows.

Proof of Theorem 1 (i). We may assume that $u \geq 0$. Let $\mathcal{E} = \{(x,y) : u(x,y) > \lambda\}$ and $E = \mathcal{E}^*$. We apply Lemmas 2, 4 and Theorem 2 (ii) to obtain

$$\begin{aligned} C_{K,p}(\{x : M_\Omega u(x) > \lambda\}) &= C_{K,p}\left(\bigcup_{(x,y)\in\mathcal{E}} (x - \Omega(y)) \right) \leq C_{K,p}(\widetilde{E}_\Omega) \\ &\leq AC_{K,p}(E) = AC_{K,p}(\{x : Nu(x) > \lambda\}). \end{aligned}$$

The theorem is proved.

3. Proof of Theorem 1 (ii) and Theorem 2 (ii)

In this section we shall prove Theorem 2 (ii) and use it to prove Theorem 1 (ii). The proof is essentially the same as in [4, Section 4]. However, for the completeness we shall give a complete proof. Let us first observe the following covering property of the Nagel-Stein approach regions. This is proved by [18, Lemma 1]. For the convenience of reader we shall give a simple proof.

Lemma 5. *Suppose that Ω satisfies the cone condition. Then the following statements are equivalent.*

(i) *Ω satisfies the cross section condition.*
(ii) *There is a positive integer N such that for any $r > 0$ there are points $x_1, \dots, x_N$ such that $\Omega(r) \subset \bigcup_{j=1}^N B(x_j, r)$.*

Proof. It is sufficient to show (i) $\Longrightarrow$ (ii). First we claim

$$B(x, r/2) \subset \Omega((1 + (2\alpha)^{-1})r) \quad \text{if } x \in \Omega(r). \tag{3.1}$$

Suppose $x \in \Omega(r)$. By the cone condition we see that $(x', r') \in \Omega$ if $|x' - x| < \alpha(r' - r)$. Let $r' = (1 + (2\alpha)^{-1})r$. Then this means $|x' - x| < r/2 \Longrightarrow x' \in \Omega((1 + (2\alpha)^{-1})r)$. Thus (3.1) follows.

Let us consider groups of points $\{x_1, \dots, x_N\} \subset \Omega(r)$ with $|x_i - x_j| \geq r$ for $i \neq j$. Then $B(x_j, r/2)$ are disjoint balls included in $\Omega((1+(2\alpha)^{-1})r)$ by the claim. Hence (i) implies that the number N is bounded by a constant independent of r. Now we choose $\{x_1, \dots, x_N\}$ with maximal N. Then the maximality implies that for any $x \in \Omega(r)$, there is x_j such that $|x - x_j| < r$, i.e. $x \in B(x_j, r)$. Hence $\Omega(r) \subset \bigcup_{j=1}^N B(x_j, r)$. The lemma follows.

Since Λ_h is an outer capacity, i.e., $\Lambda_h(E) = \inf_{E\subset V, V \text{ is open}} \Lambda_h(V)$, we may assume that E is an open set V. Let us consider a Whitney decomposition of V, i.e. Q_k are closed cubes with sides parallel to the axes with the following properties:

(i) $\bigcup Q_k = V$;
(ii) the interiors of Q_k are mutually disjoint;
(iii)

$$\operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, V^c) \leq 4 \operatorname{diam}(Q_k) \tag{3.2}$$

([19, Theorem 1 on p.167]). Let $\widetilde{Q}_k$ be the cube which has the same center as Q_k but is expanded by the factor 9/8. Then

$$\text{the multiplicity of } \widetilde{Q}_k \text{ is bounded by } N_1, \tag{3.3}$$

where N_1 depends only on the dimension n ([19, Proposition 3 on p.169]). In view of (3.2) we can choose a constant c_0, $0 < c_0 < 1$, with the property that

$$B(x, c_0\delta_V(x)) \cap Q_k \neq \emptyset \implies B(x, c_0\delta_V(x)) \subset \widetilde{Q}_k. \tag{3.4}$$

Using these facts, we can prove the following lemma.

Lemma 6. *Let h satisfy (1.4). Suppose V is a proper open subset of $\mathbb{R}^n$. Then there is a covering $\mathcal{B} = \{B(x_j, r_j)\}$ of V such that*

$$r_j \geq \delta_V(x_j), \tag{3.5}$$

$$\sum_j h(r_j) \leq A\Lambda_h(V), \tag{3.6}$$

where $A > 0$ depends only on the dimension n and h.

Proof. Since V is an open set, it follows that $\Lambda_h(V) > 0$. By definition we can find a covering $\{B(\xi_j, \rho_j)\}$ of V such that

$$\sum_j h(\rho_j) \leq 2\Lambda_h(V). \tag{3.7}$$

From this covering we construct a covering $\mathcal{B}$ with the required properties.

Let $\bigcup_k Q_k$ be the Whitney decomposition of V and let $\widetilde{Q}_k$ be the expanded cube as before the lemma. We let

$$\mathcal{K}_1 = \{k : \text{ there is } B(\xi_j, \rho_j) \text{ meeting } Q_k \text{ such that } \rho_j \geq c_0\delta_V(\xi_j)\},$$
$$\mathcal{K}_2 = \{k : \text{ if } B(\xi_j, \rho_j) \text{ meets } Q_k, \text{ then } \rho_j < c_0\delta_V(\xi_j)\},$$

where c_0 is the constant appearing in (3.4).

First suppose $k \in \mathcal{K}_1$. We can find $j = j(k)$ such that $B(\xi_j, \rho_j) \cap Q_k \neq \emptyset$ and $\rho_j \geq c_0\delta_V(\xi_j)$. Let $\xi \in B(\xi_j, \rho_j) \cap Q_k$. We have from (3.2)

$$\operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, V^c) \leq \delta_V(\xi) \leq \delta_V(\xi_j) + \rho_j \leq (1 + c_0^{-1})\rho_j.$$

Hence $Q_k \subset B(\xi_j, (2 + c_0^{-1})\rho_j)$, so that

$$\bigcup_{k\in\mathcal{K}_1} Q_k \subset \bigcup_{k\in\mathcal{K}_1} B(\xi_{j(k)}, (2 + c_0^{-1})\rho_{j(k)}), \tag{3.8}$$

$$(2 + c_0^{-1})\rho_{j(k)} \geq (2 + c_0^{-1})c_0\delta_V(\xi_{j(k)}) \geq \delta_V(\xi_{j(k)}). \tag{3.9}$$

Second suppose $k \in \mathcal{K}_2$. Since $\rho_j < c_0\delta_V(\xi_j)$ for $B(\xi_j,\rho_j) \cap Q_k \neq \emptyset$, we obtain from (3.4) that

$$Q_k \subset \bigcup_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} B(\xi_j,\rho_j) \subset \widetilde{Q}_k.$$

In particular, $\rho_j \le \dfrac{9}{16}\operatorname{diam}(Q_k)$. From (1.3) and the first inclusion we have

$$\begin{aligned}|Q_k| \le A \sum_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} \rho_j^n &= A|Q_k| \sum_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} \left(\frac{\rho_j}{\operatorname{diam}(Q_k)}\right)^n \\ &\le A|Q_k| \sum_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} h\left(\frac{\rho_j}{\operatorname{diam}(Q_k)}\right),\end{aligned}$$

so that (1.4) and the second inclusion yield

$$h(\operatorname{diam}(Q_k)) \le A \sum_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} h(\rho_j) \le A \sum_{B(\xi_j,\rho_j)\subset \widetilde{Q}_k} h(\rho_j).$$

Hence

$$\sum_{k\in\mathcal{K}_2} h(\operatorname{diam}(Q_k)) \le A \sum_{k\in\mathcal{K}_2} \sum_{B(\xi_j,\rho_j)\subset\widetilde{Q}_k} h(\rho_j) \le AN_1 \sum_j h(\rho_j), \tag{3.10}$$

where the last inequality follows from (3.3). Note that $Q_k \subset B(x_{Q_k}, \operatorname{diam}(Q_k))$ with x_{Q_k} being the center of Q_k. We have from (3.2)

$$\delta_V(x_{Q_k}) \le \operatorname{dist}(Q_k, V^c) + \operatorname{diam}(Q_k) \le 5\operatorname{diam}(Q_k). \tag{3.11}$$

We observe from (1.5), (3.7), (3.8) and (3.10) that

$$\mathcal{B} = \{B(\xi_{j(k)}, (2+c_0^{-1})\rho_{j(k)}) : k \in \mathcal{K}_1\} \cup \{B(x_{Q_k}, 5\operatorname{diam}(Q_k)) : k \in \mathcal{K}_2\}$$

is a covering of V and

$$\begin{aligned}\sum_{k\in\mathcal{K}_1} h((2+c_0^{-1})\rho_{j(k)}) &\le A\sum_j h(\rho_j) \le A\Lambda_h(V),\\ \sum_{k\in\mathcal{K}_2} h(5\operatorname{diam}(Q_k)) &\le A\sum_j h(\rho_j) \le A\Lambda_h(V).\end{aligned}$$

Thus (3.6) follows. We obtain from (3.9) and (3.11) that our covering $\mathcal{B}$ satisfies (3.5). The lemma is proved.

Proof of Theorem 2 (i). We may assume that E is a proper open set V. First we claim

$$-\Omega(y) \subset x - \Omega(y + \frac{2}{\alpha}|x|), \tag{3.12}$$

where α is the constant appearing in the cone condition. We may assume that $x \neq 0$. Suppose $\xi \in -\Omega(y)$. Then $(-\xi, y) \in \Omega$ and

$$|(-\xi + x) + \xi| = |x| < 2|x| = \alpha(y + \frac{2}{\alpha}|x| - y).$$

Hence the cone condition implies that $-\xi + x \in \Omega(y + 2|x|/\alpha)$, or equivalently $\xi \in x - \Omega(y + 2|x|/\alpha)$. The claim is proved.

By Lemma 6 we find a covering $\mathcal{B} = \{B(x_j, r_j)\}$ of V satisfying (3.5) and (3.6). Suppose $x \in B(x_j, r_j)$. Then $|x - x_j| < r_j$ and $\delta_V(x) \le 2r_j$ by (3.5), so that

$$-\Omega(\delta_V(x)) \subset x_j - x - \Omega(\delta_V(x) + \frac{2}{\alpha}|x - x_j|) \subset x_j - x - \Omega(A_1 r_j)$$

with $A_1 = 2 + 2/\alpha$ by (3.12). Hence $x - \Omega(\delta_V(x)) \subset x_j - \Omega(A_1 r_j)$, so that

$$\bigcup_{x \in B(x_j, r_j)} (x - \Omega(\delta_V(x))) \subset x_j - \Omega(A_1 r_j).$$

By Lemma 5 we find points $x_{j,\nu}$ $(\nu = 1, ..., N)$ such that

$$\Omega(A_1 r_j) \subset \bigcup_{\nu=1}^{N} B(x_{j,\nu}, A_1 r_j),$$

where the number N depends only on Ω. Hence by (1.5) and (3.6)

$$\begin{aligned} \Lambda_h\left(\bigcup_{x \in V} (x - \Omega(\delta_V(x))\right) &\le \Lambda_h\left(\bigcup_j \bigcup_{\nu=1}^{N} B(x_j - x_{j,\nu}, A_1 r_j)\right) \\ &\le \sum_j \sum_{\nu=1}^{N} h(A_1 r_j) \le A\Lambda_h(V). \end{aligned}$$

The theorem is proved.

Proof of Theorem 1 (ii). We may assume that $u \ge 0$. Let $\mathcal{E} = \{(x, y) : u(x, y) > \lambda\}$ and $E = \mathcal{E}^*$. We apply Lemmas 2, 4 and Theorem 2 (i) to obtain

$$\begin{aligned} \Lambda_h(\{x : M_\Omega u(x) > \lambda\}) = \Lambda_h\left(\bigcup_{(x,y) \in \mathcal{E}} (x - \Omega(y))\right) &\le \Lambda_h(\widetilde{E}_\Omega) \\ &\le A\Lambda_h(E) = A\Lambda_h(\{x : Nu(x) > \lambda\}). \end{aligned}$$

The theorem is proved.

4. Proof of Theorems 3 and 4

Proof of Theorem 3. We may assume that $u \ge 0$. Let $\mathcal{E} = \{(x,y) : u(x,y) > \lambda\}$. In the same way as in Lemma 4 we have

$$\{x \in \mathbb{R}^n : N^\eta u(x) > \lambda\} = \bigcup_{(x,y)\in\mathcal{E}} B(x,\eta(y)),$$

$$\{x \in \mathbb{R}^n : M_{\Omega^\eta} u(x) > \lambda\} = \bigcup_{(x,y)\in\mathcal{E}} (x - \Omega(\eta(y))).$$

Let $E = \bigcup_{(x,y)\in\mathcal{E}} B(x,\eta(y))$ and $(x,y) \in \mathcal{E}$. Then, by definition, $B(x,\eta(y)) \subset E$, and hence $\eta(y) \le \delta_E(x)$. By the monotonicity of $\Omega(y)$ we have $\Omega(\eta(y)) \subset \Omega(\delta_E(x))$. Therefore

$$\{x \in \mathbb{R}^n : M_{\Omega^\eta} u(x) > \lambda\} \subset \bigcup_{x\in E} (x - \Omega(\delta_E(x)) = \widetilde{E}_\Omega,$$

so that Theorem 2 implies

$$C_{K,p}(\{x : M_{\Omega^\eta} u(x) > \lambda\}) \le C_{K,p}(\widetilde{E}_\Omega) \le AC_{K,p}(E) = AC_{K,p}(\{x : N^\eta u(x) > \lambda\}),$$

$$\Lambda_h(\{x : M_{\Omega^\eta} u(x) > \lambda\}) \le \Lambda_h(\widetilde{E}_\Omega) \le A\Lambda_h(E) = A\Lambda_h(\{x : N^\eta u(x) > \lambda\}).$$

The theorem follows.

Theorem 4 will follow from the next geometrical observation.

Lemma 7. *Suppose $\Omega(y)$ is a nondecreasing set function of y. If $E \subset \mathbb{R}^n$, then*

$$\widetilde{E}_{\Omega^\eta} \subset \bigcup_{x\in E} (x - \Omega(\delta_{\widetilde{E}^\eta}(x))).$$

Proof. Let $\xi \in \widetilde{E}_{\Omega^\eta}$. Then there is $x \in E$ such that $\xi \in x - \Omega(\eta(\delta_E(x)))$. By definition $B(x,\eta(\delta_E(x))) \subset \widetilde{E}^\eta$ and hence $\eta(\delta_E(x)) \le \delta_{\widetilde{E}^\eta}(x)$. By the monotonicity of $\Omega(y)$ we have $\xi \in x - \Omega(\delta_{\widetilde{E}^\eta}(x))$. The lemma follows.

Proof of Theorem 4. We may assume that E is an open set. Then, by definition, $E \subset \widetilde{E}^\eta$ and hence by Lemma 7

$$\widetilde{E}_{\Omega^\eta} \subset \bigcup_{x\in\widetilde{E}^\eta} (x - \Omega(\delta_{\widetilde{E}^\eta}(x))).$$

Apply Theorem 2 with E replaced by $\widetilde{E}^\eta$. We have

$$C_{K,p}(\widetilde{E}_{\Omega^\eta}) \le C_{K,p}\left(\bigcup_{x\in\widetilde{E}^\eta} (x - \Omega(\delta_{\widetilde{E}^\eta}(x)))\right) \le AC_{K,p}(\widetilde{E}^\eta),$$

$$\Lambda_h(\widetilde{E}_{\Omega^\eta}) \le \Lambda_h\left(\bigcup_{x\in\widetilde{E}^\eta} (x - \Omega(\delta_{\widetilde{E}^\eta}(x)))\right) \le A\Lambda_h(\widetilde{E}^\eta).$$

The theorem follows.

5. Proof of Theorem 5

Proof of Theorem 5. Without loss of generality we may assume that E is open. Then by Lemma 6 we obtain a covering $\mathcal{B} = \{B(x_j, r_j)\}$ satisfying (3.5) and (3.6). By definition

$$\widetilde{E}^\eta \subset \bigcup_j \bigcup_{x \in B(x_j, r_j)} B(x, \eta(\delta_E(x)).$$

Since $\delta_E(x) \le \delta_E(x_j) + r_j \le 2r_j$ for $x \in B(x_j, r_j)$ by (3.5), it follows that

$$\widetilde{E}^\eta \subset \bigcup_j B(x_j, \eta(2r_j) + r_j) \subset \bigcup_j B(x_j, 2\eta(2r_j)),$$

where we have used $\eta(t) \ge t$ and the monotonicity. By Theorem B and the countable subadditivity

$$C_{K,p}(\widetilde{E}^\eta) \le \sum_j \Phi(2\eta(2r_j)).$$

It is easy to see that Φ satisfies the doubling condition. By assumption and the doubling property of h we have $\Phi(2\eta(2r_j)) \le A\Phi(\eta(2r_j)) \le Ah(2r_j) \le Ah(r_j)$, so that by (3.6)

$$C_{K,p}(\widetilde{E}^\eta) \le \sum_j Ah(r_j) \le A\Lambda_h(E).$$

The theorem follows.

6. Proof of Theorem 6

The proof of the theorem is based on the duality theorem of Meyers [16, Theorem 14], Kerman-Sawyer's inequality [13, (2.7) and (2.8)] and Adams' proof [1, Theorem 3.2] of Wolff's inequality [12]. Using these results, we have the following lemma.

Lemma 8. *Let*

$$W^\mu_{K,q}(x) = \int_0^\infty \mu(B(x,r))^{q-1}\overline{K}(r)^q r^n \frac{dr}{r}, \ \mathcal{W}^\mu_{K,q}(x) = \int_0^1 \mu(B(x,r))^{q-1}\overline{K}(r)^q r^n \frac{dr}{r}.$$

Then $\|K * \mu\|_q^q \le A \int W^\mu_{K,q}(x) d\mu(x)$ *for any measure* μ *and*

$$C_{K,p}(E) \ge A \sup\{\|\mu\|^p : \mu \text{ is concentrated on } E, \int W^\mu_{K,q} d\mu \le 1\}.$$

Moreover, if $\int_0^\infty K(t) t^{n-1} dt < \infty$*, then* $\|K * \mu\|_q^q \le A \int \mathcal{W}^\mu_{K,q}(x) d\mu(x)$ *and*

$$C_{K,p}(E) \ge A \sup\{\|\mu\|^p : \mu \text{ is concentrated on } E, \int \mathcal{W}^\mu_{K,q} d\mu \le 1\}.$$

For these accounts we refer to [3] and [6, Part II].

Proof of Theorem 6. The proof is a generalization and simplification of [4, Proof of Theorems 2 and 2']. In the same way as in [4], without loss of generality, we may assume that E is a bounded open set. Let F be an arbitrary compact subset of $\widetilde{E}^\eta$. it is sufficient to show that

$$\Lambda_h(F) \le AC_{K,p}(E). \tag{6.1}$$

By the Frostman lemma (cf. [8, Theorem 1 on p. 7] and [11, Lemma 5.4]) we can find a measure μ on F such that $\|\mu\| \approx \Lambda_h(F)$ and

$$\mu(B(x,r)) \le h(r) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0. \tag{6.2}$$

By definition, for each $x \in \widetilde{E}^\eta$, there is $x^* \in E$ such that $x \in B(x^*, \eta(\delta_E(x^*)))$. We let

$$r(x) = \sup_{\substack{x^* \in E \\ x \in B(x^*, \eta(\delta_E(x^*)))}} \delta_E(x^*).$$

We observe that $r(x)$ is a positive bounded function on $\widetilde{E}^\eta$. We see that $r(x)$ is a bounded positive function on $\widetilde{E}^\eta$. By $C(x,r)$ we denote the closed ball with center at x and radius r. By the Besicovitch covering lemma (see e.g. [22, Theorem 1.3.5], we can find $\{x_j\} \subset F$ such that

$$F \subset \bigcup C(x_j, 2\eta(r_j)) \text{ with } r_j = r(x_j), \tag{6.3}$$

$$\text{the multiplicity of } \{C(x_j, 2\eta(r_j))\} \text{ is bounded by } N. \tag{6.4}$$

By definition we can find $x_j^* \in E \cap B(x_j, \eta(r_j))$ such that

$$r_j/2 < \delta_E(x_j^*) \le r_j. \tag{6.5}$$

Let $\mu_j = \mu|_{C(x_j, 2\eta(r_j))}$ and observe that

$$\mu \le \sum \mu_j \le N\mu. \tag{6.6}$$

From μ_j we construct a measure λ_j by

$$\lambda_j(S) = \mu_j\left(\frac{4\eta(r_j)}{r_j}(S - x_j^*) + x_j\right) \quad \text{for Borel sets } S.$$

It is easy to see that

$$\lambda_j \text{ is concentrated on } C(x_j^*, r_j/2), \tag{6.7}$$

$$\|\lambda_j\| = \|\mu_j\| \le Ah(\eta(r_j)), \tag{6.8}$$

$$\lambda_j(B(x,\rho)) = \mu_j(B(x,\rho)) = \|\mu_j\| \quad \text{for } \rho \ge \max\{|x - x_j| + 2\eta(r_j), |x - x_j^*| + r_j/2\}. \tag{6.9}$$

In (6.8) we have used (6.2) and the doubling property of h. Moreover, we have

$$\lambda_j(B(x,r)) \le Ah\left(\frac{\eta(r_j)}{r_j}r\right) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0\ . \tag{6.10}$$

In view of (6.5) and (6.7) we see that λ_j is concentrated on E. Let $\lambda = \sum \lambda_j$. Then λ is a measure concentrated on E. We claim

$$W_{K,q}^{\lambda}(x) \le A \quad \text{for } x \in \mathbb{R}^n. \tag{6.11}$$

If this is shown, then, by (6.6), $\int W_{K,q}^{\lambda} d\lambda \le A\,\|\lambda\| \approx \|\mu\| \approx \Lambda_h(F)$ and hence by Lemma 8

$$C_{K,p}(E) \ge A\left\|\frac{\lambda}{(A\Lambda_h(F))^{1/q}}\right\|^p \approx \Lambda_h(F),$$

so that the arbitrariness of F yields the theorem.

Now let us prove (6.11). Hereafter, we fix $x \in \mathbb{R}^n$. First we claim

$$W_{K,q}^{\lambda_j}(x) \le A \tag{6.12}$$

with A independent of j and x. By (6.10) and change of the variable we have

$$\int_0^{r_j} \lambda_j(B(x,r))^{q-1}\overline{K}(r)^q r^n \frac{dr}{r} \le A\int_0^1 h(\eta(r_j)t)^{q-1}\overline{K}(r_jt)^q(r_jt)^n\frac{dt}{t},$$

where the last integral is bounded by (1.7). On the other hand, (6.8) yields

$$\begin{aligned}\int_{r_j}^{\infty} \lambda_j(B(x,r))^{q-1}\overline{K}(r)^q r^n \frac{dr}{r} &\le Ah(\eta(r_j))^{q-1}\int_{r_j}^{\infty}\overline{K}(r)^q r^n\frac{dr}{r}\\ &= A[h(\eta(r_j))/\Phi(r_j)]^{q-1},\end{aligned}$$

where the last term is bounded by assumption. Hence (6.12) follows.

Let us write

$$\lambda' = \sum{}'\lambda_j, \quad \lambda'' = \sum{}''\lambda_j,$$

where $\sum'$ (resp. $\sum''$) denotes the summation over j for which $x \in C(x_j, 2\eta(r_j))$ (resp. $x \notin C(x_j, 2\eta(r_j))$). In view of (6.4) the number of j appearing in $\sum'$ is at most N. Hence by (6.12)

$$W_{K,q}^{\lambda'}(x) \le A. \tag{6.13}$$

In order to treat $W_{K,q}^{\lambda''}(x)$, we estimate $\lambda''(B(x,r)) = \sum''\lambda_j(B(x,r))$. In the summation $\sum''$, we may consider only j such that $\lambda_j(B(x,r)) > 0$. Since λ_j is concentrated on $C(x_j^*, r_j/2)$, it follows that $|x - x_j^*| \le r + r_j/2$. On the other hand, the definition of $\sum''$ implies $|x - x_j| > 2\eta(r_j)$. Since $x_j^* \in B(x_j, \eta(r_j))$, it follows that

$$r + \eta(r_j)/2 \ge r + r_j/2 \ge |x - x_j^*| \ge |x - x_j| - |x_j - x_j^*| > 2\eta(r_j) - \eta(r_j),$$

whence $r > \eta(r_j)/2 \ge r_j/2$, $|x - x_j^*| < 2r$, $|x_j - x_j^*| < 2r$ and $|x - x_j| < 4r$. Hence $\max\{|x - x_j| + 2\eta(r_j), |x - x_j^*| + r_j/2\} < 8r$. Therefore (6.9) implies

$$\begin{aligned}\lambda''(B(x,r)) &= \sum{}''\lambda_j(B(x,r)) \le \sum{}''\lambda_j(B(x,8r)) \\ &= \sum{}''\mu_j(B(x,8r)) \le \sum \mu_j(B(x,8r)) \le N\mu(B(x,8r)) \le Ah(r),\end{aligned}$$

where we have used (6.6) and the doubling property of h. Since (1.6) follows from (1.7), we have

$$W_{K,q}^{\lambda''}(x) \le A \int_0^\infty h(r)^{q-1}\overline{K}(r)^q r^n \frac{dr}{r} < \infty.$$

This, together with (6.13), yields (6.11). The proof is complete.

Remark. In view of Lemma 8 and the above proof, (1.7) can be replaced by

$$\int_0^{\min\{1,1/r\}} h(\eta(r)t)^{q-1}\overline{K}(rt)^q (rt)^n \frac{dt}{t} \le A \quad \text{for } r > 0,$$

if $\int_0^\infty K(t)t^{n-1}dt < \infty$. This is the case when $K(t) = g_\alpha(t)$, the Bessel kernel.

Proof of Corollaries 1 and 2. We may assume that $u \ge 0$. Let $\mathcal{E} = \{x : u(x,y) > \lambda\}$ and $E = \mathcal{E}^* = \bigcup_{(x,y)\in\mathcal{E}} B(x,y)$. In view of Lemma 4 and the proof of Theorem 3 $E = \{Nu > \lambda\}$ and $\{N^\eta u > \lambda\} = \bigcup_{(x,y)\in\mathcal{E}} B(x,\eta(y))$. By definition $\delta_E(x) \ge y$ for $(x,y) \in \mathcal{E}$ and hence $\{N^\eta u > \lambda\} \subset \widetilde{E}^\eta = \bigcup_{x\in E} B(x,\eta(\delta_E(x)))$. This inclusion and Theorems 5 and 6 readily yield the first assertions of Corollaries 1 and 2. The second assertions follow from Theorem 3. The corollaries are proved.

7. Proof of Theorem 7

Proof of Theorem 7. We may assume that E is a compact set. By the Frostman lemma we find a measure μ on E such that $\Lambda_h(E) \approx \|\mu\|$ and $\mu(B(x,r)) \le h(r)$ for all $x \in \mathbb{R}^n$ and $r > 0$. We have from Lemma 8

$$\begin{aligned}\|K*\mu\|_q^q &\le A\int W_{K,q}^\mu d\mu = A\int d\mu(x)\int_0^\infty \mu(B(x,t))^{q-1}\overline{K}(t)^q t^n \frac{dt}{t} \\ &\le A\,\|\mu\| \int_0^\infty \min\{h(t), \|\mu\|\}^{q-1}\overline{K}(t)^q t^n \frac{dt}{t}.\end{aligned}$$

Using the duality theorem of Meyers [16, Theorem 14], we obtain

$$C_{K,p}(E) \ge A\,\|\mu\| \left(\int_0^\infty \min\{h(t), \|\mu\|\}^{q-1}\overline{K}(t)^q t^n \frac{dt}{t}\right)^{1-p}.$$

We observe from (1.5) that the right hand side is greater than

$$
\begin{aligned}
&A\left(\int_0^{h^{-1}(\Lambda_h(E))}\left(\frac{h(t)}{\Lambda_h(E)}\right)^{q-1}\overline{K}(t)^q t^n\frac{dt}{t}+\int_{h^{-1}(\Lambda_h(E))}^{\infty}\overline{K}(t)^q t^n\frac{dt}{t}\right)^{1-p}\\
&\ge A\left(\int_{h^{-1}(\Lambda_h(E))}^{\infty}\overline{K}(t)^q t^n\frac{dt}{t}\right)^{1-p}\\
&= A\Phi(h^{-1}(\Lambda_h(E)).
\end{aligned}
$$

Thus the theorem is proved.

Example. Let $K(t)$ be the Bessel kernel $g_\alpha(t)$. The capacity $C_{K,p}$ is called the Bessel capacity of index (α,p) and denoted by $B_{\alpha,p}$. If $0<\alpha p<n$, then $B_{\alpha,p}$ and $R_{\alpha,p}$ are locally comparable, i.e., $B_{\alpha,p}(E)\approx R_{\alpha,p}(E)$ for $E\subset B(0,R_0)$. By Theorem 7 $\Lambda_\beta(E)^{1/\beta}\le AB_{\alpha,p}(E)^{1/n-\alpha p}$ for $E\subset B(0,R_0)$ if $n-\alpha p<\beta\le n$.

If $\alpha p=n$, then $\Phi(r)\approx(\log\frac{1}{r})^{1-p}$ for $0<r<1/2$. Condition (1.8) reads as

$$\int_0^r h(t)^{q-1}\frac{dt}{t}\le Ah(r)^{q-1}\log\frac{1}{r}$$

for $0<r<1/2$. Letting $h(t)=t^\beta$ with $0<\beta\le n$, we obtain $B_{\alpha,p}(E)\ge A(\log\frac{A}{\Lambda_\beta(E)})^{1-p}$ for $E\subset B(0,R_0)$. Letting $h(t)=(\log\frac{1}{t})^{-\ell}$ for $0<t<1/2$ with $\ell>p-1$, we obtain $B_{\alpha,p}(E)\ge A\Lambda_h(E)^{(p-1)/\ell}$ for $E\subset B(0,R_0)$.

Remark. Maz'ya [15, §8.5] gave some similar results.

8. Proof of Propositions 1 and 2

Proof of Proposition 1. In view of the proof of Theorem 1, we see that (ii) $\Longrightarrow$ (i). Hence we assume (i). Let $r>0$ and let $u=P\chi_{B(0,r)}$. Then we see that $u>A_0$ on $T(B(0,r))$ (see the proof of Theorem 2 (i)). Hence, Lemmas 2 and 3 yield

$$\bigcup_{(x,y)\in T(B(0,r))}(x-\Omega(y))\subset\{x:M_\Omega u(x)>A_0\}.$$

In particular, $-\Omega(r)\subset\{x:M_\Omega u(x)>A_0\}$. On the other hand, $\{x:Nu(x)>A_0\}\subset B(0,A_2r)$ by inspection. Hence (i) implies

$$
\begin{aligned}
\Lambda_h(\Omega(r))=\Lambda_h(-\Omega(r))&\le\Lambda_h(\{M_\Omega u>A_0\})\\
&\le A\Lambda_h(\{Nu>A_0\})\le A\Lambda_h(B(0,A_2r))\le Ah(r),
\end{aligned}
$$

so that there are (x_j, r_j) such that $\Omega(r) \subset \bigcup B(x_j, r_j)$ and $\sum h(r_j) \le Ah(r)$. We may assume that $r_j \le r$. Then by (1.4) and (1.3)

$$A \ge \sum \frac{h(r_j)}{h(r)} \ge A \sum h\left(\frac{r_j}{r}\right) \ge A \sum \left(\frac{r_j}{r}\right)^n.$$

Hence $|\Omega(r)| \le A \sum r_j^n \le Ar^n$. Thus Ω satisfies the cross section condition.

Proof of Proposition 2. In the same way as in the above proof we may assume (i). Let $r > 0$ and let $u = P\chi_{B(0,r)}$. We have

$$\begin{aligned} R_{\alpha,p}(\Omega(r)) &= R_{\alpha,p}(-\Omega(r)) \le R_{\alpha,p}(\{M_\Omega u > A_0\}) \\ &\le AR_{\alpha,p}(\{Nu > A_0\}) \le AR_{\alpha,p}(B(0, A_2 r)) \le Ar^{n-\alpha p}. \end{aligned}$$

By Theorem 7 with $\beta = n$, we have $|\Omega(r)| \le Ar^n$. Thus Ω satisfies the cross section condition.

Remark. Proposition 2 cannot be generalized to the Bessel capacity $B_{\alpha,p}$ with $\alpha p = n$. Since $B_{\alpha,p}(B(0,r)) \approx (\log 1/r)^{1-p}$ as $r \to 0$, it follows that $B_{\alpha,p}(B(0,r^\varepsilon)) \approx B_{\alpha,p}(B(0,r))$ as $r \to 0$ for any $\varepsilon > 0$. Hence $B_{\alpha,p}(B(0,r^\varepsilon)) \le B_{\alpha,p}(B(0,A_2 r))$. On the other hand if $0 < \varepsilon < 1$, then $|B(0,r^\varepsilon)|/r^n \to \infty$ as $r \to 0$.

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